18/02 - Stochastic Processes and White Noise

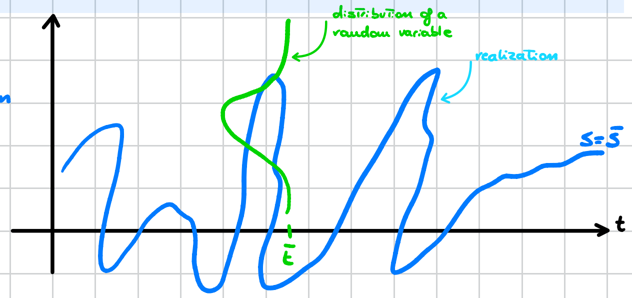
DEF: A stochastic process (SP) is an infinite sequence of random variables, all defined on the same probabilistic space

$\dots\ v(1,s)\ ,\ v(2,s)\ ,\ v(3,s)\dots$

Time index — random experiment realization

Setting S means set the whole sequence

The definition extends the notion of "random variable" to signals



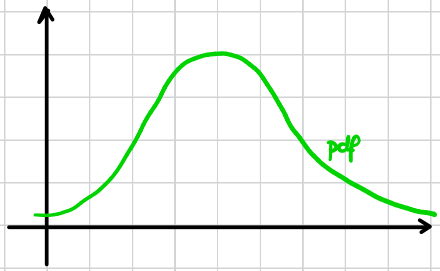
Even if $v_1(t), v_2(t), v_3(t)$ are different signal, if they are realization of the same stocastic process $v(t,s)$, they are said to be stochastically equivalent (generated by the same SP, not identical!)

deterministic signal $v(t,\bar{s})$

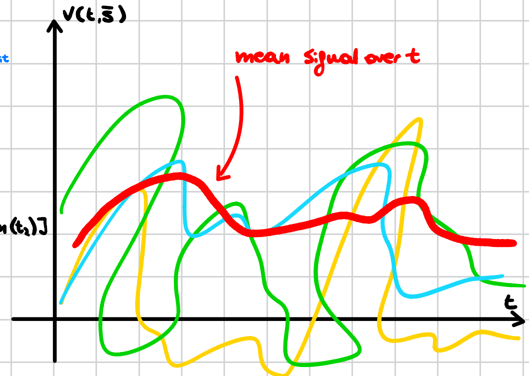
random variable $v(\bar{t},s)$

To fully describe a SP we will need to know how the probability distribution of $v(s)$ evolves over the time $t \Rightarrow$ TOO DIFFICULT FOR MODELING PURPOSE.

From now on, we'll work with the so called WIDE-SENSE CHARATERIZATION of SP (mean/variance only)



This is a gaussian, so with WIDE-SENSE CHARACTERIZATION we could work only with σ and η, with some approximation to simplify the counts (simpler than working with the pdf)



expectation — don't need to remember it

DEF: The mean value of a SP : $m(t) = E[v(t,s)] = \int_S v(t,s)\, pdf(s)\ ds$

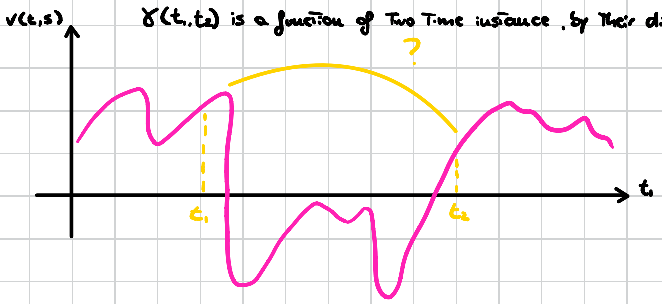
represent the average behavior of the SP

DEF: The covariance function of a SP : $\gamma(t_1,t_2) = E[v(t_1,s)-m(t_1)][v(t_2,s)-m(t_2)]$

ex: $t_1 = t_2 \Rightarrow \gamma(t_1,t_2) = \gamma(t,t) = E[(v(t,s)-m(t))^2]$ variance

$t_1 \neq t_2 \Rightarrow$ see the def.

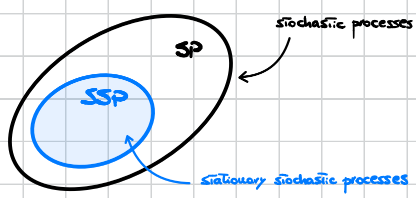
$\gamma(t_1,t_2)$ is a function of two time instance, by their distance



$v[v_1,v_2]\quad E[(v_1-m)^2]\quad E[(v_2-m)^2]$

Could be useful to know to make prediction on the future t'

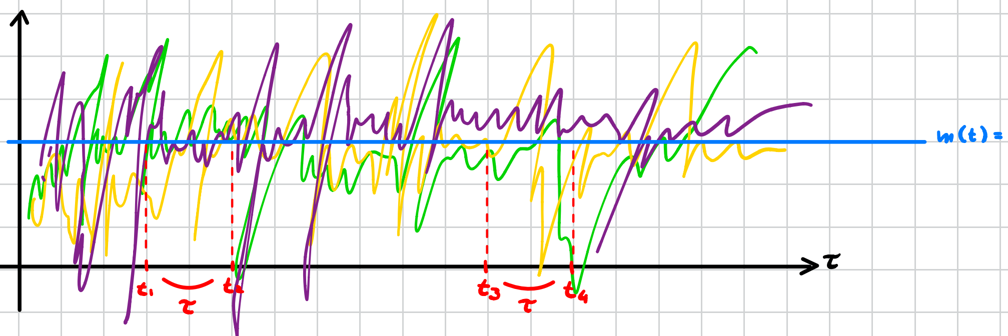
Mechatronic work with stationary stochastic process



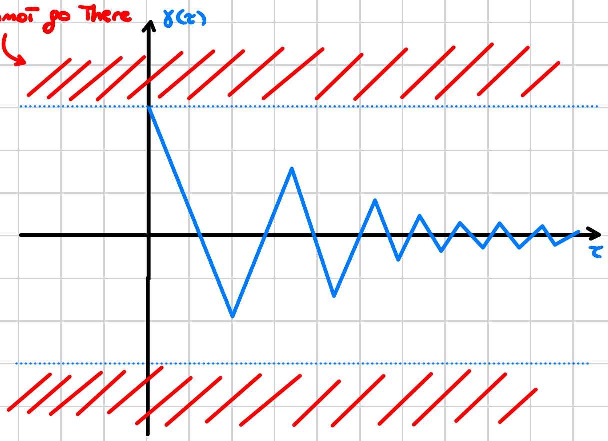
DEF: A stationary stochastic process (SSP) is a SP with

1) $m(t) = m\quad \forall t$ constant

2) $\gamma(t_1,t_2) = \gamma(t_1, t_1+\tau) = \gamma(\tau)$ this does't depend on T_1. Means that $\gamma(t_1,t_2) = \gamma(t_3,t_4)$ if $t_4 - t_3 = t_2 - t_1$ even if $t_1 \neq t_2 \neq t_3 \neq t_4$



we cannot go there



Property of covariance $\gamma(\tau)$

1) $\gamma(0) = E[(v(t,s)-m)^2] \geq 0$ by construction — Positivity
2) $|\gamma(\tau)| \leq \gamma(0) \quad \forall \tau$ — Non-increasing
3) $\gamma(\tau) = \gamma(-\tau) \quad \forall \tau$ — even function

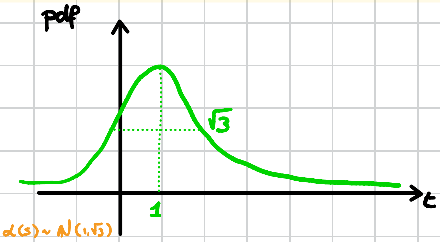
moving to the future, the correlation decreases

Observe

1) given an SSP $v(t,s)$, we will write $m_v(t)$ (or m_v) and $\gamma_v(t_1,t_2)$ (or $\gamma_v(\tau)$) to indicate its mean and covariance function
2) Two SSP $v_1(t,s)$ and $v_2(t,s)$ are Wide-sense equivalent if $m_{v_1} = m_{v_2}$ and $\gamma_{v_1}(\tau) = \gamma_{v_2}(\tau) \quad \forall \tau$
3) the covariance function $E[(v_1(t,s)-m)(v_2(t,s)-m)]$ is different from the correlation function $E[v_1(t,s) \cdot v_2(t,s)]$ if $m \neq 0$

Ex 1: $v(t,s) = \alpha(s)$ with $\alpha(s) \sim N(1,\sqrt{3})$ (so α is a gaussian variable).

Is the process stationary? Yes



Fixing s, $\alpha(s)$ became a number, so all the realization are constant

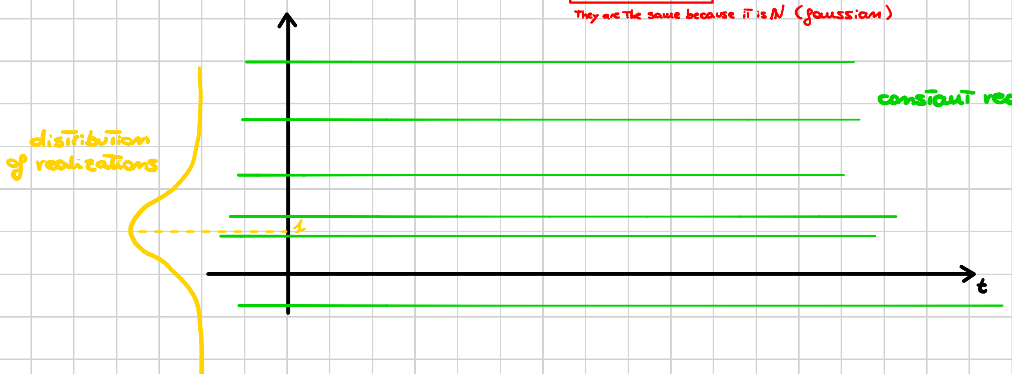
1) constant realization
2) distributed like $\alpha(s)$

by definition of $\alpha(s) \sim N(1,\sqrt{3})$

The mean is a constant: $m = E[v(t,s)] = E[\alpha(s)] = 1 \quad \forall t$

the covariance: $\gamma(t_1,t_2) = E[(v(t_1,s)-1)(v(t_2,s)-1)] = E[(\alpha(s)-1)^2] = 3 \quad \forall t, \tau$

They are the same because it is N (gaussian)



Because both m_v and γ_v do not depend on t (they are constant), the process is SSP

Ex 2: $v(t,s) = t\alpha(s) - t$ with $\alpha(s) \sim N(1,\sqrt{3})$. Is the process stationary? No

- $m(t) = E[v(t,s)] = E[t\alpha(s) - t] = E[t\alpha(s)] - E[t] = tE[\alpha(s)] - E[t] = t \cdot 1 - t = 0$ constant $\forall t$

  E is a linear operator

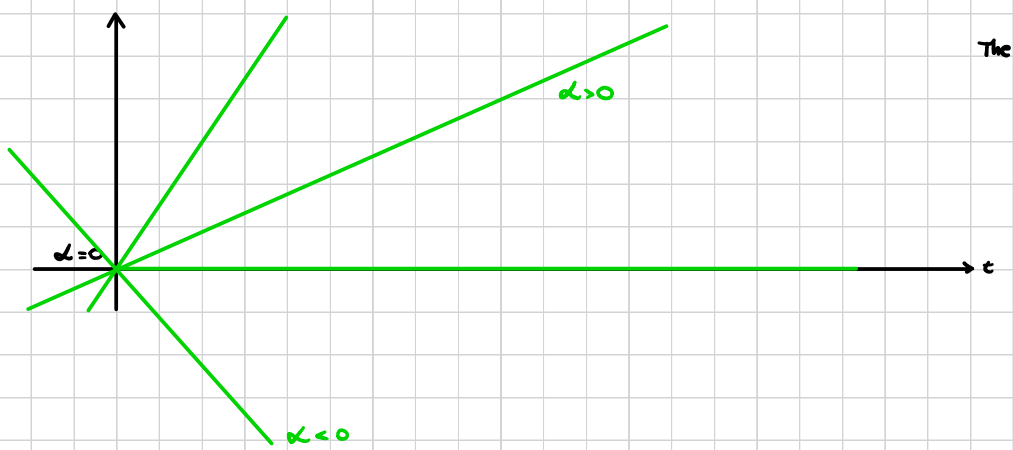
- $\gamma(t_1,t_2) = E[(v(t_1,s))(v(t_2,s))] = E[(t_1\alpha(s)-t_1)(t_2\alpha(s)-t_2)]$ $\quad t_2 = t_1+\tau \quad t_1 = t$

$= E[(t\alpha(s)-t)((t+\tau)\alpha(s)-(t+\tau))]$

$= E[t(\alpha(s)-1)(t+\tau)(\alpha(s)-1)]$

$= t(t+\tau)\underbrace{E[(\alpha(s)-1)^2]}_{3}$

$= 3t(t+\tau)$ not verified because depends on t



The variance is not constant!

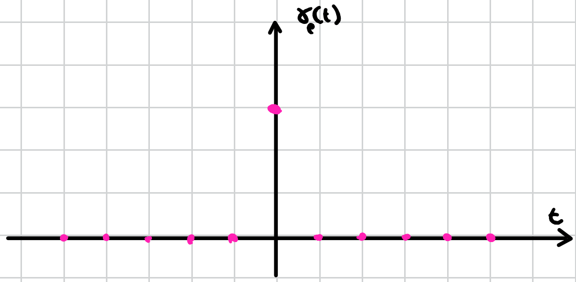
White Noise

A SSP $e(t)$ is called white noise (WN) with mean μ and variance λ^2 if the following holds:

a) $E[e(t)] = \mu$ mean value

b) $E[(e(t)-\mu)^2] = \lambda^2$ variance

c) $\gamma_e(\tau) = E[(e(t)-\mu)(e(t+\tau)-\mu)] = 0 \quad \forall t \neq 0$ covariance



We will write $e(t) \sim WN(\mu, \lambda^2)$ to define a white noise signal

Is an unpredictable process (0 correllation on the future)

A constant signal $e(t)$ like the one represented here is possible?

This signal $e(t) \sim WN(\mu, \lambda^2)$ is the so so called constant realization and is one of the possible (but very unlikely) realization of the WN. We don't know the pdf.