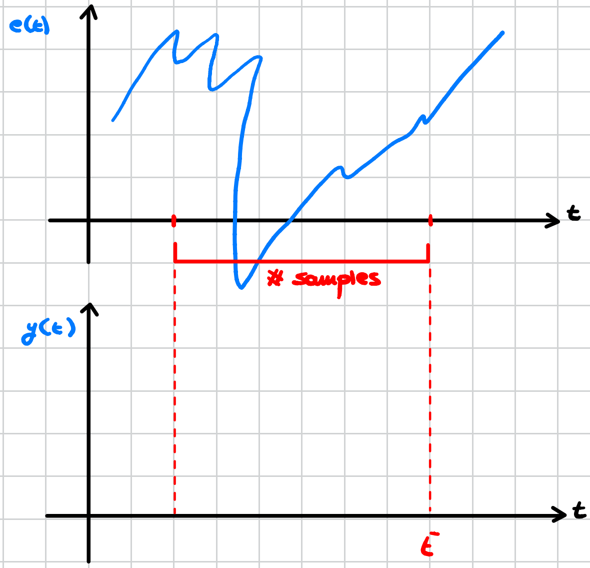
19/02 - Moving Average

DEF: Let $e(t) \sim WN(0, \lambda^2)$ (for semplicity $\mu = 0$) and consider the process obtained as
$y(t) = c_0 e(t) + c_1 e(t-1) + \dots + c_n e(t-n)$ where c_i are the weights and $y(t)$ is called **MOVING AVERAGE PROCESS** of order n; and it is denoted as $MA(n)$



$y(\bar{t})$ is a function of $e(\bar{t}) \dots e(\bar{t}-n)$

Is $MA(n)$ a stationary process?

Is sochastic for sure because $e(t) \sim WN(0, \lambda^2)$

$$
\begin{aligned}
- m_y(t) &= E[y(t)] \\
&= E[c_0 e(t) + \dots + c_n e(t-n)] \\
&= c_0 E[e(t)] + \dots + c_n E[e(t-n)] \quad \text{because } E \text{ is linear} \\
&= (c_0 + \dots + c_n) \underline{E[e(t)]}_{0} \\
&= \sum_{i=0}^{n} c_i m_e = 0 \quad \text{constant}
\end{aligned}
$$

and so we can write $m_y(t) = m_y = 0 \quad \forall t$

- $\tau = 0$

$$
\begin{aligned}
- \gamma_y(t,t) &= E[(y(t) - m_y)^2] = E[y^2(t)] \quad \text{Simplified because } m_y = 0 \\
&= E[(c_0 e(t) + \dots + c_n e(t-n))^2] \\
&= E[c_0^2 e^2(t) + \dots + c_n^2 e^2(t-n) + 2 c_0 c_1 e(t) e(t-1) + \dots + 2 c_{n-1} c_n e(t-n+1) e(t-n)] \\
&= E[c_0^2 e^2(t)] + \dots + E[2 c_{n-1} c_n e(t-n+1) e(t-n)] \\
&= c_0^2 E[e^2(t)] + \dots + c_n^2 E[e^2(t-n)] + \cancel{2 c_0 c_1 E[e(t) e(t-1)]} + \dots
\end{aligned}
$$

Remember: $E[e^2(t)] = E[(e(t) - m)^2] \hat{=} \lambda^2 \quad \forall t$ — $e(t)$ is stationary

$E[e(t-i) e(t-j)] = 0 \quad \forall i,j;\ i \neq j$ — $e(t)$ is white noise

→ in two different time instant (t and $t-1$) the two sample of white noise are uncorrelated

$$
= c_0^2 \lambda^2 + c_1^2 \lambda^2 + \dots + c_n \lambda^2 = \sum_{i=0}^{n} c_i \lambda^2 \quad \text{constant}
$$

- $\tau = 1$

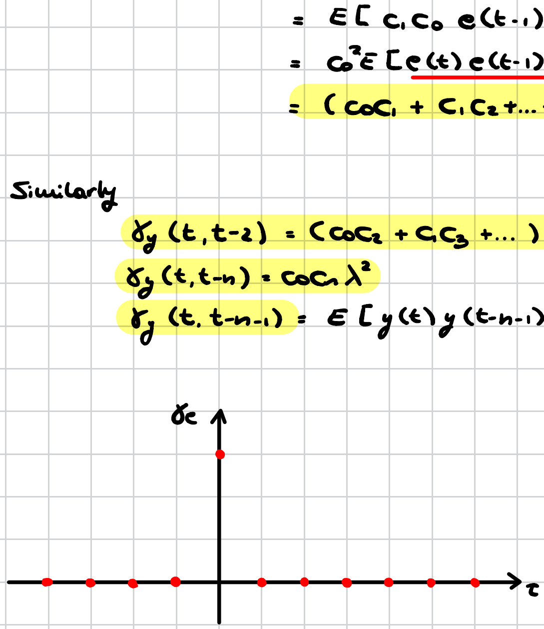
$$
\begin{aligned}
- \gamma_y(t,t-1) &= E[(y(t) - m_y)(y(t-1) - m_y)] = E[y(t) y(t-1)] \\
&= E[(c_0 e(t) + \dots + c_n e(t-n))(c_0 e(t-1) + \dots + c_n e(t-n-1))] \\
&= E[c_0^2 e(t) e(t-1) + c_0 c_1 e(t) e(t-2) + \dots + c_1 c_0 e(t-1)^2 + c_n e(t-1) e(t-2) + \dots] \\
&= E[c_1 c_0 e(t-1)^2] + E[c_2 c_1 e(t-2)^2] + \dots + E[c_0^2 e(t) e(t-1)] + \dots \\
&= c_0^2 \underline{E[e(t) e(t-1)]}_{0} + c_0 c_1 \underline{E[e(t) e(t-2)]}_{0} + \dots + c_0 c_1 \underline{E[e(t-1)^2]}_{\lambda^2} + c_1^2 \underline{E[e(t-1) e(t-2)]}_{0} \\
&= (c_0 c_1 + c_1 c_2 + \dots + c_n c_{n-1}) \lambda^2 \quad \text{constant}
\end{aligned}
$$

Similarly

$\gamma_y(t, t-2) = (c_0 c_2 + c_1 c_3 + \dots) \lambda^2$ — $n-1$ element in the sum

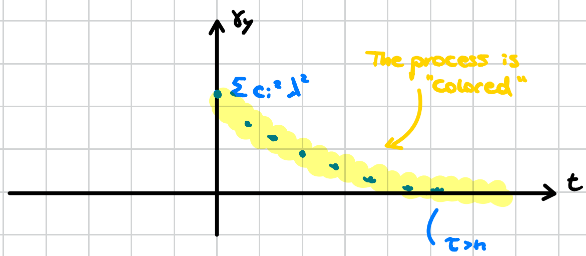
$\gamma_y(t, t-n) = c_0 c_n \lambda^2$ — only 1 element in the sum of coefficient

$\gamma_y(t, t-n-1) = E[y(t) y(t-n-1)] = E[(c_0 e(t) + \dots + c_n e(t-n))(c_0 e(t-n-1) + \dots)] = 0 \quad \forall \tau > n$



$c_0, \dots, c_n$

=>



This is only a mathematical object, WN do not exists!

Generalization of MA(n) : $y(t) = \sum_{i=0}^{+\infty} c_i e(t-i)$ with $e(t) \sim WN(0,\lambda^2)$

Moving Average Processes of infinite grade (MA(∞))

An MA(∞) process is a generalization of MA(n) and it can be written as

$$y(t) = \sum_{i=0}^{+\infty} c_i e(t-i) \qquad e(t) \sim WN(0,\lambda^2)$$

Assumption: $\sum_0^{+\infty} c_i^2 < +\infty$

Mean: $m_y(t) = E[y(t)] = E[\sum c_i e(t-i)] = \sum c_i E[e(t-i)] = \sum c_i 0 = 0$

Variance:
$$\begin{aligned}\gamma_y(t,t) &= E[y(t)^2] = E[(\sum c_i e(t-i))(\sum c_j e(t-j))] \\ &= E[\sum_{i,j} c_i c_j e(t-i) e(t-j)] \\ &= \sum c_i c_j E[e(t-i)e(t-j)] = \sum c_i^2 \lambda^2 < +\infty\end{aligned}$$

Covariance
$$\begin{aligned}\gamma_y(t,t-\tau) &= E[\sum c_i e(t-i) \sum c_j e(t-j-\tau)] \\ &= E[\sum c_i c_j e(t-i) e(t-j-\tau)] \\ &= \sum_{i,j=0}^{+\infty} c_i c_j \underbrace{E[e(t-i)e(t-j-\tau)]}_{\neq 0 \text{ only if } i=j+\tau} = \sum_{i=0}^{+\infty} c_i c_{i+\tau} \lambda^2\end{aligned}$$

MA(∞) is a SSP that can model any process that we will see in this course