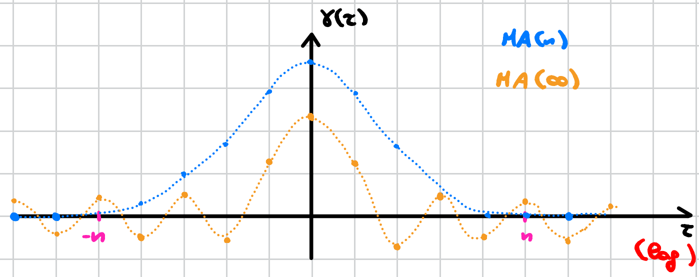
24/02 - Autoregressive Models

The main weak point of MA is related to the choice of τ. $MA(\tau)$ is practical but limited modeling power; $MA(\infty)$ can model any SSP, but not usable in practice (∞ coefficient) due to its heavy computation.



There exist a class of SSP that can shape $\gamma(\tau)$ $\forall \tau$ with a finite set of coefficient: **Autoregressive Models**.

DEF: Let $e(t) \sim WN(0,\lambda^2)$ and consider $y(t) = a_1 y(t-1) + a_2 y(t-2) \dots + a_m y(t-m) + e(t)$. Then $y(t)$ is said **AUTOREGRESSIVE MODEL** of order m ($AR(m)$), with m be the number of coefficient.

$y(t)$ depends on time and on its previous outcome ($y(t-1), y(t-2), \dots$)

AR $\Leftrightarrow$ steady-state solution of difference equation. We don't care about the initial condition, we only consider the relationship with the input

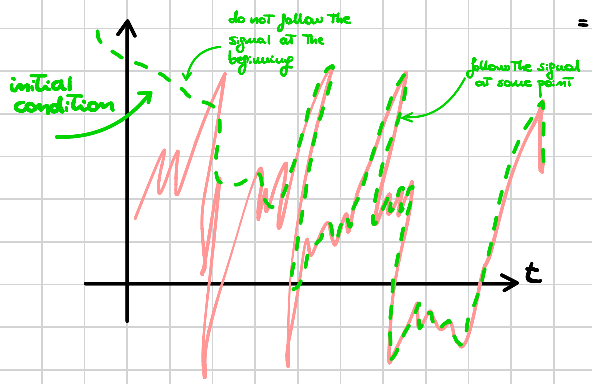
example: Consider the process $y(t) = a\,y(t-1) + e(t)$ with $e(t) \sim WN(0,\lambda^2)$

Steady-state solution:

$$\begin{aligned} y(t) &= a\,y(t-1) + e(t) && \text{write } y(t-1) = a y(t-2) + e(t-1) \\ &= a\,(a\,y(t-2) + e(t-1)) + e(t) \\ &= a^2 y(t-2) + e(t) + a\,e(t-1) && \text{write } y(t-2) = a y(t-3) + e(t-2) \\ &= a^2(a\,y(t-3) + e(t-2)) + e(t) + a\,e(t-1) \\ &\vdots \end{aligned}$$

I can continue forever, until...

$$= \underbrace{e(t) + a\,e(t-1) + a^2 e(t-2) + \dots}_{\text{external excitation}} + \underbrace{a^{t-t_0} y(t_0)}_{\text{initial condition (do not consider)}} \quad \text{with } t_0 \to -\infty$$



For Autoregressive Model (of SSP) with CANNOT consider the initial condition component in the solution $\Rightarrow$ **The initial part is not SSP**

The external excitation is a linear combination that is a $MA(\infty)$ (even with a single coefficient)

So, $AR(1) \cong MA(\infty)$ with $c_1 = 1, c_2 = a, \dots$. This is why we shall obtain $\gamma(\tau) \neq 0$ also for $\tau \to +\infty$

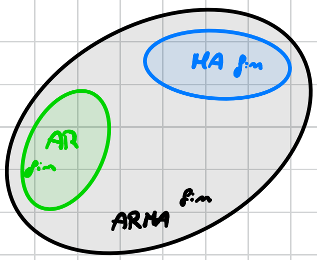
Remark: The modeling power of $MA(\infty)$ is still higher than that of $AR(1)$, as the coefficient are constrained.

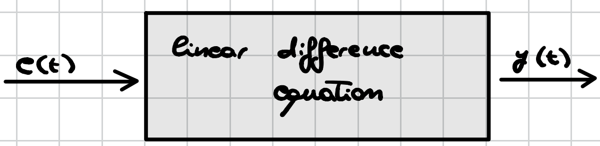
ARMA models (Autoregressive + Moving Average models)

DEF: Let $e(t) \sim WN(0,\lambda^2)$ and consider a process $y(t)$ made by an $AR(m)$ and a $MA(n)$ parts:

$$y(t) = \underbrace{a_1 y(t-1) + \dots + a_m y(t-m)}_{AR(m)} + \underbrace{c_0 e(t) + \dots + c_n e(t-n)}_{MA(n)}$$

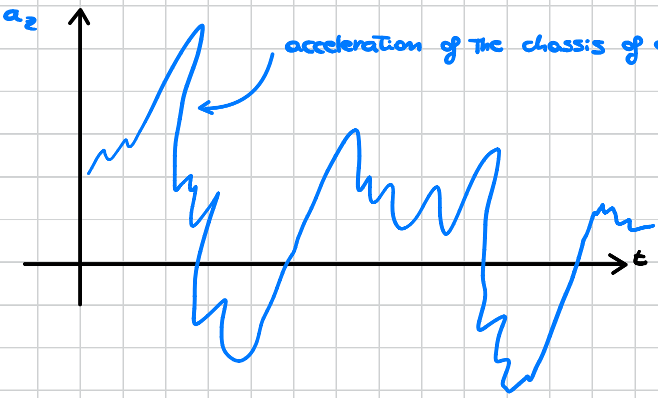
$y(t)$ is said **ARMA(m,n)**, with m be the number of coefficient of the $AR(m)$ part, and n the same for $MA(n)$.

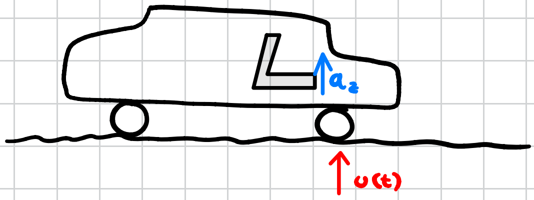


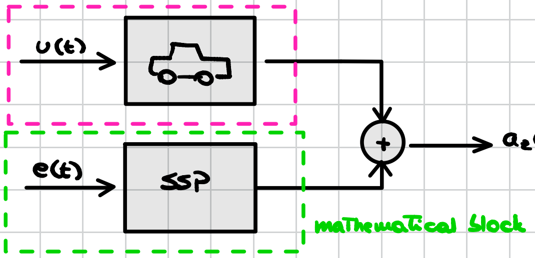


ARMA are powerful for modeling time series

In practice, consider for example the acceleration of a car...







Combination of a physical phenomena and a mathematical block, makes $a_z(t)$ a SSP

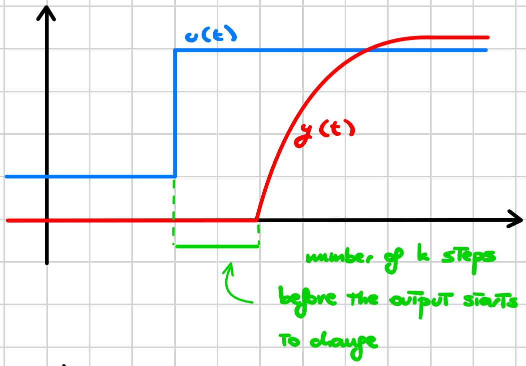
ARMAX models (ARMA + Exogenous part)

An ARMAX model adds to the classic ARMA model an exogenous component that represents all the external input of the process. So an ARMAX model is defined as:

$$
\begin{aligned}
y(t) = \; & a_1 y(t-1) + \dots + a_m (t-m) + && AR(m)\\
& c_0 e(t) + \dots + c_n e(t-n) + && MA(n)\\
& b_0 u(t-k) + \dots + b_p u(t-k-p) && X(k,p)
\end{aligned}
$$

where

- k: pure input-output delay
- p: order of exogenous part



N-ARMAX model (nonlinear)

$$y(t) = f(\, y(t-1), \dots, y(t-m),\; e(t), \dots, e(t-n),\; u(t-k), \dots u(t-k-p))$$

where the function f could be:

polynomial   spline   wavelets   neural networks

Is the AR model stationary? We need additional tools to study the stationarity of models including AR parts

Operatorial Representation

DEF: · The backward-shift operator z^{-1} is defined as $z^{-1}x(t) = x(t-1)$

· The forward-shift operator z is defined as $z\,x(t) = x(t+1)$

PROPERTIES:

- Linearity: $z^{-1}(a\,x(t) + b\,y(t)) = z^{-1}a\,x(t) + z^{-1}b\,y(t)$
- Recursive application: $z^{-1}(z^{-1}(z^{-1}(x(t)))) = z^{-1}(z^{-1}(x(t-1))) = \dots = z^{-3}x(t) = x(t-3)$
- Linear composition: $(a z^{-1} + b z + c z^{-3} + d z^{2})\,x(t) = a\,x(t-1) + b\,x(t+1) + c\,x(t-3) + d\,x(t+2)$

given an ARMA(m,n) model:

$$y(t) = a_1 y(t-1) + a_2 y(t-2) + \dots + c_0 e(t) + \dots + c_n e(t-n)$$

we can apply z^{-1} (backward-shift operator):

$$y(t) = a_1 z^{-1} y(t) + a_2 z^{-2} y(t) + \dots + c_0 e(t) + \dots + c_n z^{-n} e(t)$$

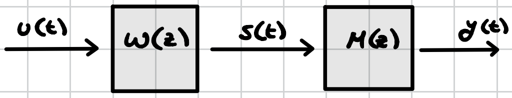
$$\Rightarrow \quad (1 - a_1 z^{-1} - a_2 z^{-2} - \dots - a_m z^{-m})\, y(t) = (c_0 + c_1 z^{-1} \dots + c_n z^{-n})\, e(t)$$

$$y(t) = \underbrace{\frac{c_0 + c_1 z^{-1} + \dots + c_m z^{-n}}{1 - a_1 z^{-1} - \dots - a_m z^{-m}}}_{\text{Transfer function}}\, e(t) = \frac{C(z)}{A(z)}\, e(t) = \underbrace{W(z)\, e(t)}_{\text{discrete-time transfer function}}$$

Operations with Transfer function

Let $W(z)$ and $H(z)$ be Transfer function of linear digital filters

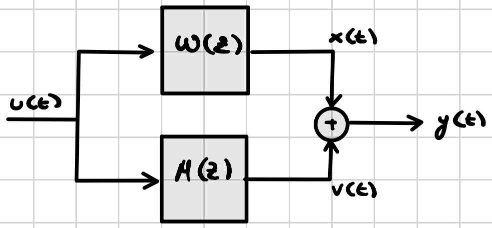
Series



$$y(t) = H(z)\, s(t)$$
$$s(t) = W(z)\, u(t)$$

$$y(t) = \underbrace{H(z)W(z)}_{\text{product}}\, u(t)$$

Parallel



$$y(t) = x(t) + v(t) = W(z)\, u(t) + H(z)\, u(t)$$

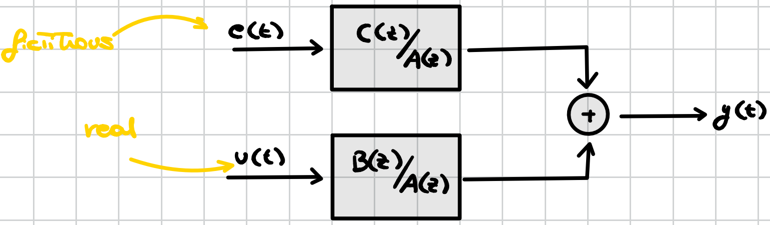
$$y(t) = \underbrace{(W(z) + H(z))}_{\text{sum}}\, u(t)$$

Considering an ARMAX model now

$$y(t) = a_1 y(t-1) + \dots + a_m y(t-m) \quad \Rightarrow a_1 z^{-1} y(t) + \dots + a_m z^{-m} y(t)$$
$$+ c_0 e(t) + \dots + c_n e(t-n) \quad \Rightarrow c_0 e(t) + \dots + c_n z^{-n} e(t)$$
$$+ b_0 u(t-k) + \dots + b_p u(t-k-p) \quad \Rightarrow b_0 z^{-k} u(t) + b_1 z^{-k} z^{-1} u(t) + \dots + b_p z^{-k} z^{-p} u(t)$$

$$\underbrace{(1 - a_1 z^{-1} - \dots - a_m z^{-m})}_{A(z)}\, y(t) = \underbrace{(c_0 + c_1 z^{-1} + \dots + c_n z^{-n})}_{C(z)}\, e(t) + \underbrace{(b_0 + b_1 z^{-1} + \dots + b_p z^{-p})}_{B(z)}\, z^{-k} u(t)$$

$$\Rightarrow y(t) = \frac{B(z)}{A(z)} z^{-k} u(t) + \frac{C(z)}{A(z)} e(t) \qquad e(t) \sim WN(0, \lambda^2)$$



$u(t)$ is "real", whereas $e(t)$ is fictitous

! $\dfrac{c_0 + c_1 z^{-1}}{1 - a z^{-1}} = \dfrac{c_0 z + c_1}{z - a_1}$ both the negative / positive power of z will be used but please don't mix them!

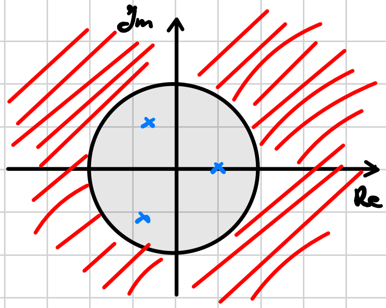
Poles / zeros

$$W(z) = \frac{C(z)}{A(z)}$$

zeros : roots of NUM

poles : roots of DEN (in case of no cancelation)

THEOREM: A linear digital filter with transfer function $W(z)$ is asymptotically stable if and only if all its poles are strictly inside the unit circle (in the complex plane)



THEOREM: The steady-state output $y(t)$ is stationary if and only if

a) $e(t)$ is SSP ($e(t) \sim WN(0, \lambda^2)$ in our scenario, so it's always true)

b) $W(z)$ is asymptotically stable