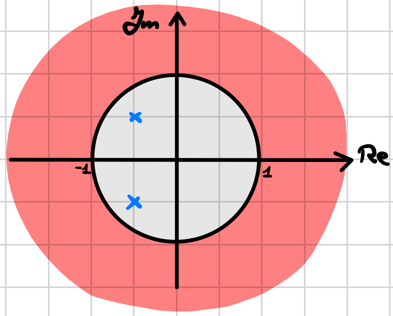
25/02 - Mean and covariance function of ARMA models

AR (MA) process represents a SSP if and only if the Transfer function $W(z)$ is asymptotically stable (poles of $W(z)$ located inside the unary circle in the complex plane). So

$$W(z) = \frac{C(z)}{A(z)} \Rightarrow A(z) = 0 \Rightarrow |z_1,..| < 1$$



Example 1: consider $y(t) = \frac{1-\frac{1}{2}z^{-1}}{1+\frac{1}{3}z^{-1}} e(t)$ with $e(t) \sim WN(0,\lambda^2)$

$$w(z) = \frac{1-\frac{1}{2}z^{-1}}{1+\frac{1}{3}z^{-1}} \cdot \frac{z}{z} \Rightarrow A(z) = z + \frac{1}{3} = 0 \Rightarrow z = -\frac{1}{3} \Rightarrow |z| = |-\frac{1}{3}| < 1 \Rightarrow$$ $W(z)$ is Asymptotically stable

So $y(z)$ is a SSP. If also we have that the zeros follows the same rule of the poles, we have

$$C(z) = z - \frac{1}{2} = 0 \Rightarrow z = \frac{1}{2} \Rightarrow |\frac{1}{2}| < 1 \Rightarrow$$ $y(t)$ is Minimum phase

Example 2: consider $y(t) = e(t) + \frac{1}{2}e(t-1) + \frac{1}{4}e(t-2)$ $\quad e(t) \sim WN(0,\lambda^2)$

We need to find first of all the transfer function. Start by applying the backward-shift operator z^{-1}:

$$y(t) = (1 + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2})e(t) \Rightarrow W(z) = 1 + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2} \Rightarrow MA(2)$$

The transfer function must be in the form of $W(z) = \frac{C(z)}{A(z)}$ to see the poles. So we multiply by z^2/z^2 to see the poles:

$$W(z) = 1 + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2} \cdot z^2/z^2 = \frac{z^2 + \frac{1}{2}z + \frac{1}{4}}{z^2}$$

This is always the case (for MA processes): $z^n = 0$

This a MA process $y(t) = \frac{C(z)}{A(z)} e(t)$ that is always SSP (also proved) independently by $\frac{C(z)}{A(z)}$ coefficient (the poles are always in 0)

Important reminder
- $MA(n)$ process has always n non trivial zeros and n poles, all lying on the origin (This is also called "all-zeros" process)
- $AR(m)$ process has n non trivial poles, has n zeros all lying on the origin (This is also called "all-poles" process)

Mean and covariance function

Suppose our model is asymptotically stable (it represents a SSP), how we can compute the mean and the covariance function?

Let's start from a simple case:

$$y(t) = a y(t-1) + e(t) \qquad e(t) \sim WN(0,\lambda^2)$$

We need to assume $y(t)$ is SP and place a condition regarding a:

$$y(t) = a y(t-1) + e(t) \Rightarrow (1 - az^{-1}) y(t) = e(t) \Rightarrow y(t) = \frac{1}{1-az^{-1}} \frac{z}{z} e(t) \Rightarrow W(z) = \frac{z}{z-a}$$

where the poles are $z = a$ and so, the condition will be $|a| < 1$ in order to let $y(t)$ a SP.

I can write $y(t)$ as an $MA(\infty)$... and use the tools we already know to compute the mean and covariance function

Long Division: I have to divide $C(z)/A(z)$ as long as you want...

1	$1-az^{-1}$
$1-az^{-1}$	$1+az^{-1}...$
$0+az^{-1}$	
$-az^{-1}+a^2z^{-2}$	
$0 \quad a^2z^{-2}$	

Continue until no rest... (infinite many times)

In this way, we can write

$$y(t) = \frac{C(z)}{A(z)} e(t) = (1 + az^{-1} + ...) e(t) = E(z) e(t)$$

To obtain no rest, I need to divide C/A infinitly many times. So $E(z)$ is a polynomial of order ∞ $\Rightarrow y(t)$ is written as $MA(\infty)$

It is not practical with a non finite number of coefficient

MEAN: $E[y(t)] = m_y = E[a y(t-1) + e(t)] = a E[y(t-1)] + E[e(t)] = a E[y(t-1)] + m_e$

Due to the supposition made before ($y(t)$ is SSP) we have that $E[y(t-1)] = E[y(t)]$, and so

$$m_y = a m_y + m_e \Rightarrow m_y = \frac{1}{1-a} m_e = 0$$

where $1-a \neq 0$ due to the condition $|a|<1$

VAR: $\gamma_y(0) = E[(y(t)-m_y)^2] = E[y^2(t)]$

$$= E[(a y(t-1) + e(t))^2]$$

$$= E[a^2 y^2(t-1) + e^2(t) + 2a y(t-1) e(t)]$$

$$= a^2 E[y^2(t-1)] + \underbrace{E[e^2(t)]}_{\lambda^2} + 2a \cancel{E[y(t-1)e(t)]}^{0}$$

Proof of that later...

Being $y(t)$ a SSP, we can write $E[y^2(t-1)] = E[y^2(t)] = \gamma_y(0)$. So

$$\gamma_y(0) = a^2 \gamma_y(0) + \lambda^2 \Rightarrow \gamma_y(0) = \frac{1}{1-a^2}\lambda^2$$

rember that $|a|<1$

COV: $\gamma_y(1) = E[(y(t)-m_y)(y(t-1)-m_y)] = E[y(t)y(t-1)]$

$$= E[(a y(t-1) + e(t)) y(t-1)]$$

$$= E[a y^2(t-1) + e(t) y(t-1)]$$

$$= a \underbrace{E[y^2(t-1)]}_{\gamma_y(0)} + \cancel{E[e(t)y(t-1)]}^{0} \Rightarrow \gamma_y(1) = a\gamma_y(0)$$

$\gamma_y(2) = E[y(t)y(t-2)]$

$$= E[(a y(t-1) + e(t)) y(t-2)]$$

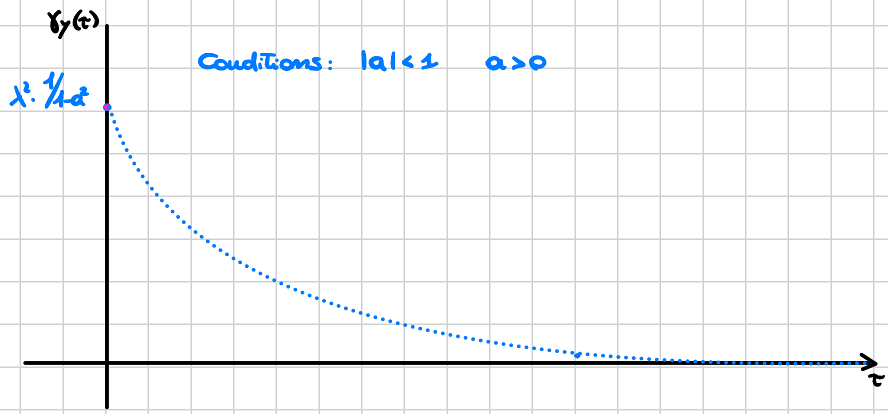
$$= E[a y(t-1) y(t-2)] + \cancel{E[e(t)y(t-2)]}^{0} \Rightarrow \gamma_y(2) = a E[y(t-1)y(t-2)] = a\gamma_y(1)$$

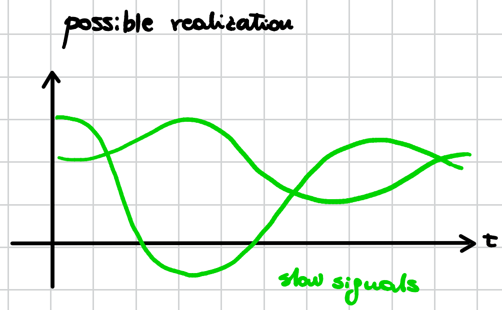
So we can intuitively find that

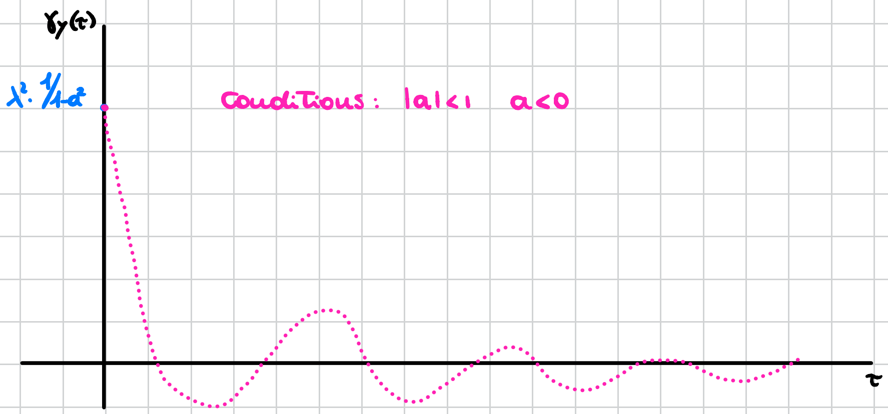
$\gamma_y(3) = a\gamma_y(2)$ $\quad \gamma_y(4) = a\gamma_y(3) \dots$ $\quad \gamma_y(\tau) = a\gamma_y(\tau-1)$ $\quad \gamma_y(0) = \frac{1}{1-a^2}\lambda^2$

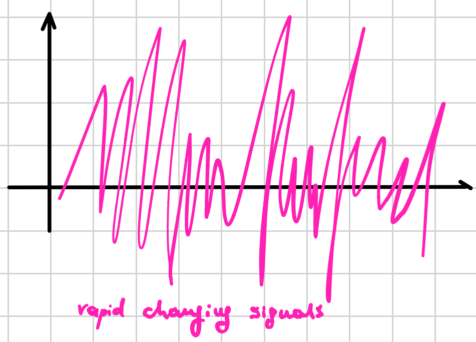
These are a recursive set of equation (in the sense that each equation depends on the previous one)

These set of equation is called Yule-Walker equation for AR(1) model









Proof: $E[e(t)y(t-\tau)] = 0$ for $\tau > 0$

Rember That $y(t) = e(t) + a\,e(t-1) + a^2 e(t-2) \dots$

Then: $y(t-1) = e(t-1) + a e(t-2) + a^2 e(t-3) \dots$ is a simple Translation in Time domain of $y(t)$

So

$$E[e(t)y(t-1)] = E[e(t)\{e(t-1) + a e(t-2) + \dots\}]$$

$$= \cancel{E[e(t)e(t-1)]} + a\,\cancel{E[e(t)e(t-1)]}^{\,0} + \dots$$

All Terms are 0 because $e(t) \sim WN(0,\lambda^2)$ and, in Two distinct Time instant, The white noise are uncorrelated.

Mean and covariance function of ARMA (n, m)

Consider The following model $y(t) = a_n y(t-1) + \dots + a_n y(t-n) + c_0 e(t) + \dots + c_m e(t-m)$ and assume That $a_1, \dots, a_n$ are such That $y(t)$ is SSP.

MEAN: $E[y(t)] = E[a_1 y(t-1) + \dots + a_n y(t-n) + c_0 e(t) + \dots + c_m e(t-m)]$

$$= a_1 \underbrace{E[y(t-1)]}_{m_y} + \dots + a_n E[y(t-n)] + c_0 \underbrace{E[e(t)]}_{m_e} + \dots + c_m E[e(t-m)]$$

$$\Rightarrow (1 - a_1 - \dots - a_n) m_y = (c_0 + \dots + c_m) m_e \quad \Rightarrow \quad m_y = \frac{c_0 + \dots + c_m}{1 - a_1 + \dots + a_n} m_e = 0$$

COVARIANCE: $\gamma_y(0) = E[y^2(t)]$

$$= E[(a_1 y(t-1) + \dots + a_n y(t-n) + c_0 e(t) + \dots + c_m e(t-m))^2]$$

$$= E[a_1^2 y(t-1)^2 + \dots + a_n^2 y(t-n)^2 + c_0^2 e^2(t) + \dots + c_m^2 e^2(t-m) +$$

$2a_1 a_2\, y(t-1) y(t-2) + \dots +$ ← $\gamma_y(\tau)$ and other γ_y's

$2a_1 c_0\, e(t) y(t-1) + \dots +$ ← combination of products of AR and MA parts

$2 c_0 c_1\, e(t) e(t-1) + \dots$ ← $=0$

$]$

$\gamma_y(0)$ depends on some $\gamma_y(\tau)$

$$\begin{cases} \gamma_y(0) = a_1^2 \gamma_y(0) + a_2^2 \gamma_y(0) + \dots + 2 a_1 a_2 \underline{\gamma_y(1)} + \dots \\ \gamma_y(1) = a_1 \gamma_y(0) + \dots + \gamma_y(1) + \dots + \gamma_y(2) + \dots \\ \vdots \\ \gamma_y(n-1) = a_1 \gamma_y(n-2) + \dots \end{cases}$$

Need To solve a system of m equation

Set of m recursive equation for The initialization for the Yule-Walker equation for ARMA (n,m)

What about m_y and $\gamma_y(\tau)$ if $e(t)$ is non-zero mean? Given $e(t) \sim WN(\mu, \lambda^2)$ with $\mu \neq 0$ (possibly)

$\tau = 0$: $E[(e(t) - \mu)^2] = \lambda^2$

$$\Rightarrow = E[e^2(t) + \mu^2 - 2\mu e(t)] = E[e^2(t)] + E[\mu^2] - 2\mu E[e(t)]$$

constant

$$= E[e^2(t)] + \mu^2 - 2\mu\mu$$

$$= E[e^2(t)] - \mu^2 = \lambda^2 \quad \Rightarrow \quad \boxed{E[e(t)^2] = \lambda^2 + \mu^2}$$ Autocorrelation

$\tau \neq 0$: $E[(e(t) - \mu)(e(t-\tau) - \mu)] = 0$

$$\Rightarrow E[e(t)e(t-\tau)] + \mu^2 - \mu E[e(t-\tau)] - \mu E[e(t)] = E[e(t)e(t-\tau)] + \mu^2 - \mu^2 - \mu^2$$

$$\Rightarrow E[e(t)e(t-\tau)] = \mu^2 \quad \forall \tau \neq 0$$

Instead of computing $\gamma_y(\tau)$ using The definition (This would be cumbersome), we define The "unbiased" version of $e(t)$ and $y(t)$ (we remove The offset)

$$\tilde{y}(t) = y(t) - m_y \qquad \tilde{e}(t) = e(t) - \mu$$

$\tilde{y}(t) = y(t) - m_y$

$= a_1 y(t-1) + \dots + a_m y(t-m) - m_y$

$= a_1(\tilde{y}(t-1) + m_y) + a_2(\tilde{y}(t-2) - m_y) + \dots - m_y$ we do The same for $e(t)$

$= a_1 \tilde{y}(t-1) + a_2 \tilde{y}(t-2) + \dots + c_0 \tilde{e}(t) + \dots + c_n \tilde{e}(t-n)$ zero-mean WN

$+ a_1 m_y + \dots + a_m m_y + c_0 \mu + \dots + c_n \mu - m_y$

$= a_1 \tilde{y}(t-1) + a_2 \tilde{y}(t-2) + \dots + c_0 \tilde{e}(t) + \dots + c_n \tilde{e}(t-n) \underbrace{- \cancel{(1-a_1-\dots-a_m) m_y} + \cancel{(c_0+\dots+c_n)\mu}}_{\text{mean value of an ARMA process}}$ => both are 0

Remember That $m_y = \dfrac{c_0+\dots+c_n}{1-a_1-\dots-a_m} m_e = \dfrac{c_0+\dots+c_n}{1-a_1+\dots+a_m}\mu$, so $(1-a_1-\dots-a_m) m_y = (c_0+\dots+c_m)\mu$

It means That we can compute $\gamma_{\tilde{y}}(t)$ as The covariance function of a process generated by a zero-mean $WN(0,\lambda^2)$

$$\gamma_{\tilde{y}}(\tau) = E[\tilde{y}(t)\tilde{y}(t-\tau)] = E[(y(t)-m_y)(y(t-\tau)-m_y)] = \gamma_y(\tau)$$

Working with The unbiased process doesn't change The measure of The variance