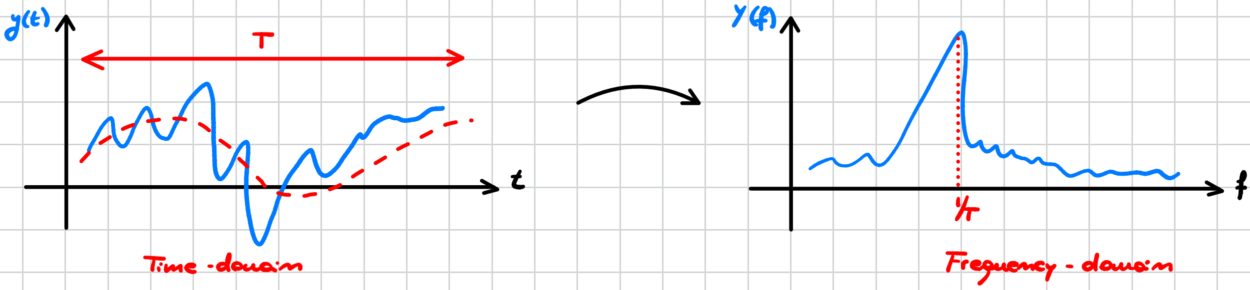
26/02 - Analysis in the frequency domain

In the deterministic world, we can transform a signal from the time domain (evolution of the signal) to the frequency domain (power of the signal). We indicate with T the period of the signal.



In the stochastic world, we have a SSP with multiple realization of s (with different time histories of the same process), so we can have different spectrum of the same process

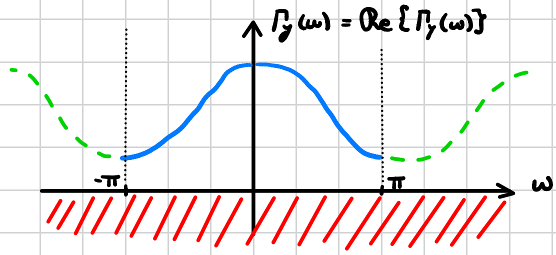
Spectral density of a SSP

DEF: The spectral density (or "spectrum") of a SSP $y(t)$ is : $\Gamma_y(\omega) = \sum_{\tau=-\infty}^{+\infty} \gamma_y(\tau) e^{-j\omega\tau}$ where ω is the frequency (j is the imaginary unit)

$\Gamma_y(\omega) = \mathcal{F}\{\gamma_y(\tau)\}$ is the Discrete Fourier Transformation (DFT) of the covariance function $\gamma_y(\tau)$ of the process $y(t)$

PROPERTIES

- $\Gamma_y(\omega)$ is a Real function of a real variable ω : $\mathrm{Im}(\Gamma_y(\omega)) = 0 \quad \forall \omega \in \mathbb{R}$ (no need to use 3D plot)
- $\Gamma_y(\omega) \geq 0 \quad \forall \omega \in \mathbb{R}$ POSITIVE
- $\Gamma_y(\omega) = \Gamma_y(-\omega) \quad \forall \omega \in \mathbb{R}$ EVEN
- $\Gamma_y(\omega) = \Gamma_y(\omega + 2\pi k) \quad \forall \omega \in \mathbb{R},\ z \in \mathbb{Z}$ 2π-PERIODIC



minimum period that can be represented : 2 samples. In fact
$\omega \leq \pi = \omega_{max}$ (Nyquist frequency)
where we can write ω as

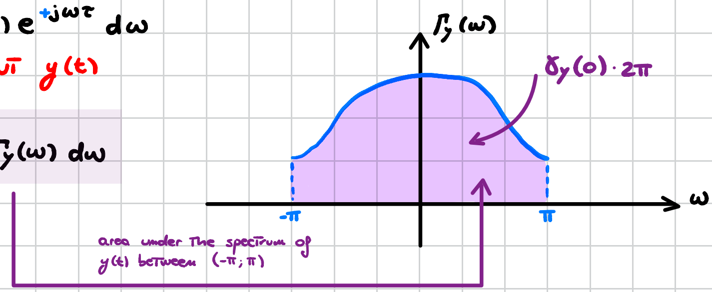
$$\frac{2\pi}{T} = \omega \leq \pi = \omega_{max} \quad \Rightarrow \quad T \geq T_{min} = \frac{2\pi}{\omega_{max}} = \frac{2\pi}{\pi} = 2$$

Inverse Transform : $\gamma_y(\tau) = \mathcal{F}^{-1}\{\Gamma_y(\omega)\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Gamma_y(\omega) e^{+j\omega\tau} d\omega$

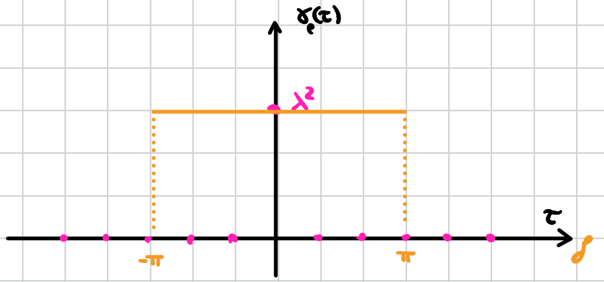
$\gamma_y(\tau)$ and $\Gamma_y(\omega)$ carry the same information about $y(t)$

Taking $\tau = 0$:

$$\gamma_y(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Gamma_y(\omega) e^{+j\omega \cdot 0} d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Gamma_y(\omega)\, d\omega$$



Start with the WHITE NOISE $e(t) \sim WN(0, \lambda^2)$ and compute its spectrum:



$$\Gamma_e(\omega) = \sum_\tau \gamma_e(\tau) e^{-j\omega\tau}$$
$$= \gamma_e(0) e^{-j\pi 0} + \cancel{\gamma_e(1) e^{-j\pi 1}}^{\,0} + \cancel{\gamma_e(-1) e^{j\pi 1}}^{\,0} + \dots \qquad \gamma_e(\tau \neq 0) = 0$$
$$= \gamma_e(0) \cdot 1 = \lambda^2$$

The spectrum of a WN is flat between $(-\pi; \pi)$

But we cannot use the definition to compute $\Gamma_y(\omega)$ for a generic ARMA $y(t)$, because we would have infinitly many $\gamma_y(\tau)$.

THEOREM of SPECTRAL FACTORIZATION

$$v(t) \longrightarrow \boxed{W(z)} \longrightarrow y(t)$$

Consider a SSP $v(t)$ as input of a asymptotically stable transfer function $W(z)$ (in this way, $y(t)$ is SSP). So the spectral density is:

$$\Gamma_y(\omega) = \Gamma_v(\omega) \left| W(z = e^{j\pi}) \right|^2$$

(if $v \sim WN(0, \lambda^2)$, $\Gamma_v(\omega) = \lambda^2$)

Remark: $W(z)\big|_{z = e^{j\omega}} = W(e^{j\omega})$ is called *frequency response* of $W(z)$

Example: Consider an MA(1) process $y(t) = e(t) + c\,e(t-1)$ with $c \in \mathbb{R}$, $e(t) \sim WN(0,1)$

Using the formula found previously we can find

$$\gamma_y(0) = (1^2 + c^2) \cdot 1 \qquad \gamma_y(1) = 1 \cdot c = c = \gamma_y(-1) \qquad \gamma_y(\tau) = 0 \quad \forall \tau > 1$$

so, the spectrum is

$$\Gamma_y(\omega) = \sum_\tau \gamma_y(\tau) e^{-j\omega\tau} = \gamma_y(0) e^{-j\omega 0} + \gamma_y(1) e^{-j\omega 1} + \gamma_y(-1) e^{j\omega 1}$$

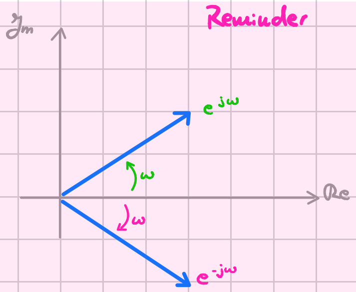
Remember also the negative value of τ

$$= (1 + c^2) \cdot 1 + c e^{-j\omega} + c e^{j\omega}$$
$$= 1 + c^2 + c(e^{-j\omega} + e^{j\omega})$$
$$= \boxed{1 + c^2 + 2c\cos(\omega)}$$

That is a REAL, EVEN, POSITIVE and 2π-PERIODIC

$1 + c^2 + 2c\cos(-1) = 1 + c^2 - 2c = (1-c)^2$

Reminder



$$e^{j\omega} = \cos(\omega) + j\,\text{sen}(\omega)$$
$$e^{-j\omega} = \cos(\omega) - j\,\text{sen}(\omega)$$
$$\Rightarrow e^{j\omega} + e^{-j\omega} = 2\cos(\omega)$$
$$\Rightarrow e^{j\omega} - e^{-j\omega} = 2j\,\text{sen}(\omega)$$

What if we use the theorem?

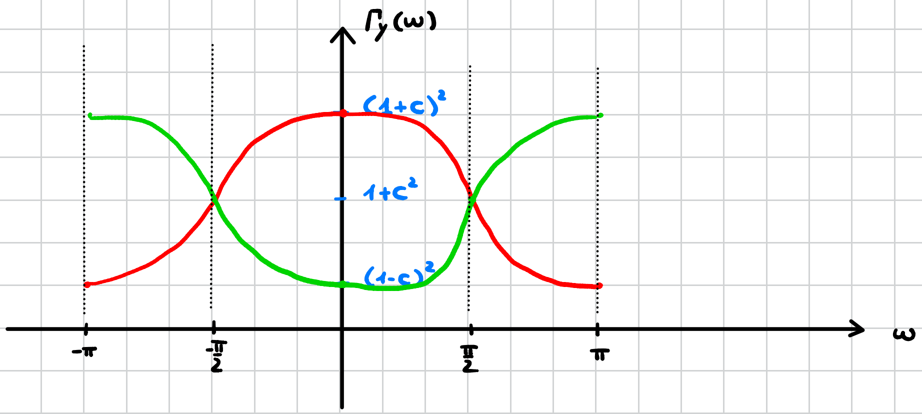
$$y(t) = e(t) + c\,e(t-1) = (1 + cz^{-1})\, e(t) = W(z)\, e(t)$$
$$\Gamma_y(\omega) = \left| W(z = e^{j\omega}) \right|^2 \underbrace{\Gamma_e(\omega)}_{\lambda^2 = 1}$$

$$\Gamma_y(\omega) = |W(e^{j\omega})|^2 = |1 + c e^{-j\omega}|^2 = (1 + c e^{-j\omega})(1 + c e^{j\omega})$$
$$= 1 + c^2 e^{-j\omega} e^{j\omega} + c e^{-j\omega} + c e^{j\omega}$$
$$= 1 + c^2 + c(e^{-j\omega} + e^{j\omega})$$
$$= \boxed{1 + c^2 + 2c\cos(\omega)}$$

same result!

For $c > 0$ $\quad (1+c)^2 > (1-c)^2$

For $c < 0$ $\quad (1+c)^2 < (1-c)^2$



Inverse process, so $\gamma_y(0)$ based on $\Gamma_y(\omega)$: we need to apply the inverse Transform $\gamma_y(\tau) = \int_{-\pi}^{\pi} \Gamma_y(\omega) e^{j\omega\tau} d\omega$

$$\gamma_y(0) = 1/2\pi \int_{-\pi}^{\pi} \Gamma_y(\omega) \cancel{e^{j\omega 0}}^{1} d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} (1+c^2+2c\cos(\omega))\, d\omega$$

$$= 1/2\pi \left\{ \left[(1+c^2)\,\omega \right]_{-\pi}^{\pi} + 2c\left[\mathrm{sen}(\omega)\right]_{-\pi}^{\pi} \right\}$$

$$= 1/2\pi \left\{ (1+c^2)\pi - (1+c^2)(-\pi) + 2c(0-0) \right\}$$

$$= 1/2\pi \left\{ (1+c^2)\, 2\pi \right\} = 1+c^2$$

Different representation of ARMA processes

1) Time-domain $\quad y(t) = a_1 y(t-1) + \dots + c_0 e(t) + \dots + c_m e(t-n)$
2) Operational $\quad y(t) = \frac{C(z)}{A(z)} e(t)$
3) Probabilistic $\quad m_y,\ \gamma_y(\tau)$
4) Frequency-domain $\quad m_y,\ \Gamma_y(\omega)$