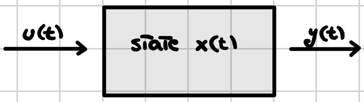
Black Box system identification using non parametric method in state space domain and time domain



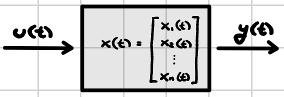
Before presenting the system identification algorithm, we recall the 3 main representations of a discrete-time, dynamic, linear systems

1- State space (SS) representation

2- Transfer function (TF) representation

3- Impulse response (IR) representation

State space representation (internal)



The description of the system is based on a vector of internal variables $x(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T$

called **states**

The system is described using two equations

$$\begin{cases} x(t+1) = Fx(t) + Cu(t) \\ y(t) = Hx(t) + Du(t) \end{cases}$$

state equation (difference equation)

output equation

If system is single-input/single-output (SISO) we can see 4 matrices:

$$F = \begin{bmatrix} n \times n \end{bmatrix} \qquad G = \begin{bmatrix} n \times 1 \end{bmatrix} \qquad H = \begin{bmatrix} 1 \times n \end{bmatrix} \qquad D = \begin{bmatrix} 1 \times 1 \end{bmatrix}$$

where n is the order of the system

If $D=0$ the system is strictly proper (the input doesn't affect directly the output)

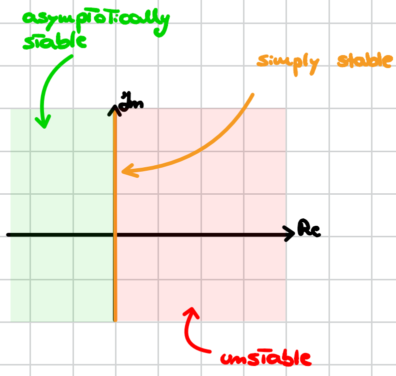
The most important property is Stability

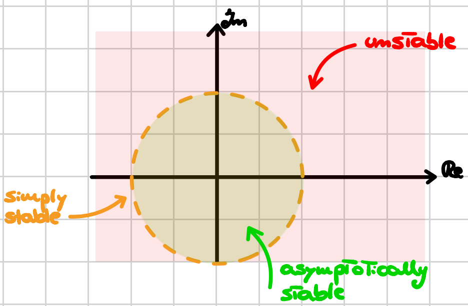
In continuous time domain, the system can be written as

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

where the stability is calculated as the real part of the Eigenvalue $EIG(A)$

- if $Re[EIG(A)] < 0$ the system is asymptotically stable
- if $Re[EIG(A)] > 0$ the system is unstable
- if $Re[EIG(A)] = 0$ the system is simply stable





In discrete time domain the system is written using the SS representation as

$$\begin{cases} x(t+1) = Fx(t) + G\,u(t) \\ y(t) = Hx(t) + Du(t) \end{cases}$$

and the stability $EIG(F)$ depends on the unit cycle in the complex plane

- if $EIG(F)$ is inside the unit cycle, the system is asymptotically stable
- if $EIG(F)$ is outside the unit cycle, the system is unstable
- if $EIG(F)$ is on the unit cycle, the system is simply stable

Example: Consider the following SISO system of order $n=2$

$$\begin{cases} x_1(t+1) = \frac{1}{2}x_1(t) + 2u(t) \\ x_2(t+1) = x_1(t) + 2x_2(t) + u(t) \\ y(t) = \frac{1}{4}x_1(t) + \frac{1}{2}x_2(t) \end{cases} \Rightarrow \quad F = \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & 2 \end{bmatrix} \quad G = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad H = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \end{bmatrix} \quad D = 0$$

unstable!

we can see that:

- F is a triangular, so $EIG(F)$ can be directly found on the main diagonal (in this case: $\frac{1}{2}$ and 2)
- 2 ($\in F$) is out the unit cycle so the system is unstable
- System is strictly proper $D=0$

State space representation is not unique: we can take $\forall T_1$ invertible matrix

$$F \to F_1 = T_1 F T_1^{-1} \qquad G \to G_1 = T_1 G \qquad H \to H_1 = H T_1^{-1} \qquad D \to D_1 = D$$

are equivalent representations

Transfer function (TF) representation (external representation)

We can easily obtain a TF representation from an I/O time domain difference equation of the system by using the delay operator z^{-1} (or the forward operator z^{+1}), that is a linear operator:

$$z^{-1}x(t) = x(t-1) \qquad z^{+1}x(t) = x(t+1)$$

Example: Start from an I/O difference equation of a signal

$$y(t) = -\tfrac{1}{6}\,y(t-1) - \tfrac{1}{8}\,y(t-2) + \tfrac{1}{2}\,u(t-1) + \tfrac{1}{4}\,u(t-2)$$

output at base (present) time t — past output (recursive part) — past input

Using z, we can bring all signals to present time

$$y(t)\left(1+\tfrac{1}{6}z^{-1}+\tfrac{1}{8}z^{-2}\right) = \left(\tfrac{1}{2}z^{-1}+\tfrac{1}{4}z^{-2}\right)u(t) \;\Rightarrow\; y(t) = \frac{\tfrac{1}{2}z^{-1}+\tfrac{1}{4}z^{-2}}{1+\tfrac{1}{6}z^{-1}+\tfrac{1}{8}z^{-2}}\,u(t)$$

negative powers representation

$$\Rightarrow\; y(t) = \frac{\tfrac{1}{2}z+\tfrac{1}{4}}{z^2+\tfrac{1}{6}z+\tfrac{1}{8}}\,u(t)$$

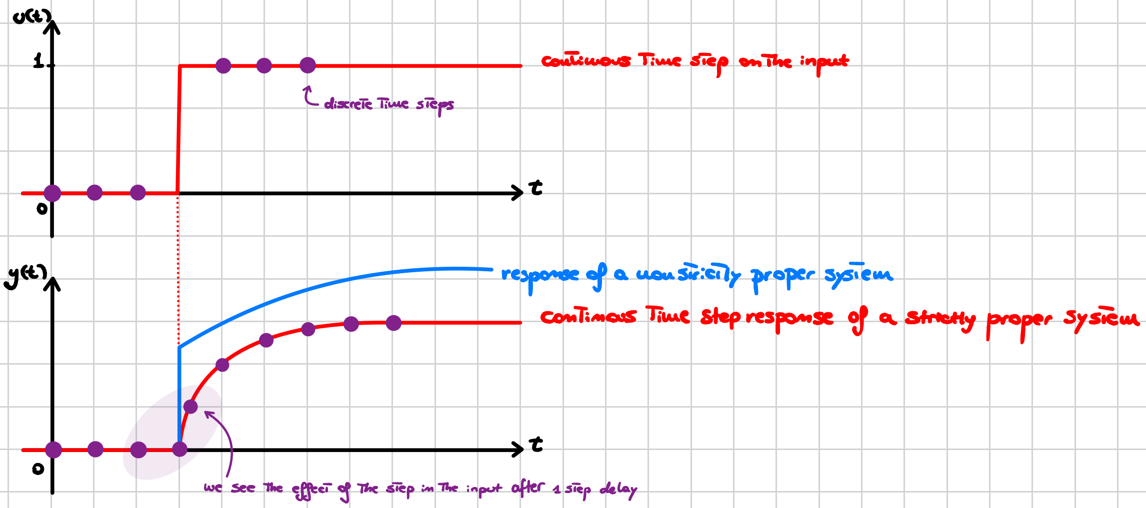
positive power representation

In general we can say that the TF description of a system is:

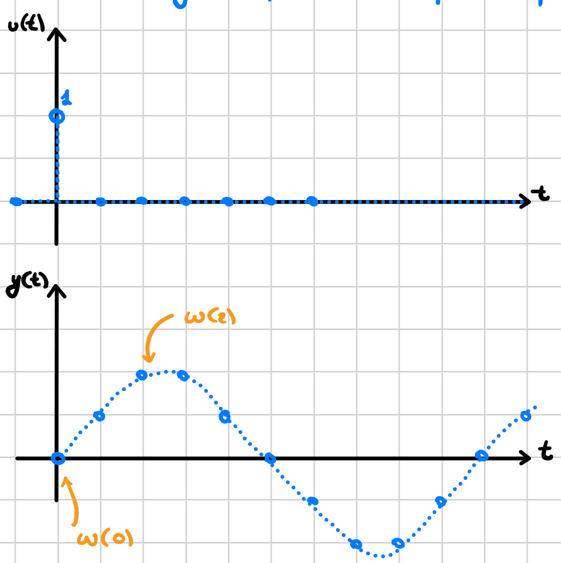
$$W(z) = \frac{B(z)}{A(z)}z^{-k} = \frac{b_0+b_1z^{-1}+\dots+b_pz^{-p}}{a_0+a_1z^{-1}+\dots+a_nz^{-n}}\,z^{-k}$$

pure delay from input to output

As we can see, DEN and NUM have the same order. If the system is strictly proper we have z^{-k} with $k \geq 1$



Convolution of the input with the impulse response (impulse response representation)



Considering an impulse signal in discrete time domain. In the continuous time domain this is the Dirac Delta $\delta(t) = 1$ for $t=0$. This is a standard conventional input. The IR response is written as

$$IR = \{w(0), w(1), \dots\}$$

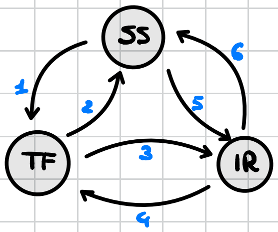
This vector of numbers provides a unique description of the system

If the system is strictly proper, we have that $w(0) = 0$

It can be proven that the output of the system can be written as

$$y(t) = \sum_{k=0}^{+\infty} w(k)\,u(t-k) = w(0)u(t) + w(1)u(t-1) + \dots$$

So the output $y(t)$ is the convolution of the input signal $u(t)$ with the IR vector



To sum up, we have 3 different representations and 6 possible transformations between pairs of representations, but... are all possible?

1. State Space to Transfer function

Consider the state space representation

$$\begin{cases} x(t+1) = Fx(t) + Gu(t) \\ y(t) = Hx(t) + \cancel{Du(t)} \end{cases}$$

using the z operator:

identity matrix $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

$$z\,x(t) = Fx(t) + Gu(t) \Rightarrow (zI-F)x(t) = Gu(t) \Rightarrow x(t) = (zI-F)^{-1}Gu(t)$$

So

Transfer function

$$y(t) = \boxed{H(zI-F)^{-1}G}\,u(t) = W(z)\,u(t) \qquad W(z) = H(zI-F)^{-1}G$$

example: consider the SISO system with 2 states $(n=2)$

$$\begin{cases} x_1(t+1) = x_1(t) + u(t) \\ x_2(t+1) = \frac{1}{2}x_1(t) + 2x_2(t) + u(t) \\ y(t) = x_1(t) \end{cases} \Rightarrow F = \begin{bmatrix}1 & 0\\ 1/2 & 2\end{bmatrix} \quad G = \begin{bmatrix}1\\1\end{bmatrix} \quad H = \begin{bmatrix}1 & 0\end{bmatrix} \quad D = 0$$

Using the formula we can find the transfer function as

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}^{-1} = \frac{1}{\det[\,]}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$$

$$W(z) = [1\ 0]\left(\begin{bmatrix}z&0\\0&z\end{bmatrix} - \begin{bmatrix}1&0\\1/2&2\end{bmatrix}\right)^{-1}\begin{bmatrix}1\\1\end{bmatrix}$$

$$= [1\ 0]\begin{bmatrix}z-1&0\\-1/2&z-2\end{bmatrix}^{-1}\begin{bmatrix}1\\1\end{bmatrix}$$

pole-zero cancelation

$$= [1\ 0]\frac{1}{(z-1)(z-2)}\begin{bmatrix}z-2&0\\1/2&z-1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \dots = \frac{\cancel{z-2}}{\cancel{(z-2)}(z-1)} = \frac{1}{z-1} = \frac{z^{-1}}{1-z^{-1}}$$

Alternative way is a direct manipulation of SS equation

$$x_1(t+1) = x_1(t) + u(t) \Rightarrow z\,x_1(t) = x_1(t) + u(t) \Rightarrow x_1(t) = \frac{1}{z-1}u(t)$$

$$x_2(t+1) = \frac{1}{2}x_1(t) + 2x_2(t) + u(t) \Rightarrow (z-2)x_2(t) = \frac{1}{2}\left(\frac{1}{z-1}u(t)\right) + u(t)$$ ⇒ useless to find $W(z)$ since...

$$y(t) = x_1(t) = \frac{1}{z-1}u(t)$$

Notice that, SS representation has order $n=2$ (so 2 eigenvalues of F), so TF representation is a system of order 1 (1 pole). Mismatch of orders because the system is not fully observable

2. Transfer function to State Space (not very used in practice) called Realization of a TF into a SS model

Issue: SS representation is not unique (∞ equivalent representation of a transfer function into an SS model)

Let's see the formula (as an example) of a special representation called control representation

$$W(z) = \frac{b_0 z^{n-1} + b_1 z^{n-2} + \dots + b_{n-1}}{z^n + a_1 z^{n-1} + \dots + a_n} \Rightarrow F = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -a_n & \dots & -a_1\end{bmatrix} \quad G = \begin{bmatrix}0\\ \vdots\\ 1\end{bmatrix} \quad H = \begin{bmatrix}b_{n-1} & \dots & b_0\end{bmatrix} \quad D = 0$$

Notice that the DEN is forced to be monic and the system is strictly proper and of order n

example: consider the transfer function of order 3

$$W(z) = \frac{2z^2 + 1/2\,z + 1/4}{z^3 + 1/4\,z^2 + 1/3\,z + 1/5} \Rightarrow F = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -\frac{1}{5} & -\frac{1}{3} & -\frac{1}{4}\end{bmatrix} \quad G = \begin{bmatrix}0\\0\\1\end{bmatrix} \quad H = \begin{bmatrix}1/4 & 1/2 & 2\end{bmatrix} \quad D = 0$$

3. Transfer function to Impulse Response

Easy via long polynomial division of the transfer function (it's easier to compute if $W(z)$ is written using the z^- notation)

example:

$$W(z) = \frac{1}{z - 1/2} = \frac{z^{-1}}{1 - 1/2\, z^{-1}}$$

$$\begin{array}{l|l} z^{-1} & 1 - 1/2\, z^{-1} \\ \hline -z^{-1} + \frac{1}{2} z^{-2} & z^{-1} + 1/2\, z^{-2} + 1/4\, z^{-3} \dots \\ \quad 1/2\, z^{-2} & \\ \quad -1/2\, z^{-2} + 1/4\, z^{-3} & \\ \qquad 1/4\, z^{-3} & \\ \qquad -1/4\, z^{-3} + 1/8\, z^{-4} & \\ \qquad\quad 1/8\, z^{-4} \dots & \end{array}$$

We can rewrite $W(z)$ as

$$W(z) = 1 \cdot z^{-1} + \frac{1}{2} z^{-2} + 1/4\, z^{-3} \dots$$

so the output signal $y(t) = W(z)\, u(t)$ will be

$$y(t) = W(z)\, u(t) = u(t-1) + \frac{1}{2} u(t-2) + 1/4\, u(t-3) + \dots$$

This is the convolution of $u(t)$ with IR:

$$IR = \{0, 1, 1/2, 1/4 \dots\}$$

The same result can be obtained in a more elegant $\quad y(t) = \left[z^{-1} \frac{1}{1 - 1/2\, z^{-1}} \right] u(t)$

Recall the geometrical series $\sum_0^{+\infty} a^k = 1/1-a$ so in our case $a = 1/2\, z^{-1}$ and

$$y(t) = \left[z^{-1} \sum_0^{+\infty} (1/2\, z^{-1})^k \right] u(t) = \dots = 0\, u(t) + u(t-1) + \frac{1}{2} u(t-2) + \frac{1}{4} u(t-3) \dots$$

Notation: name used for digital filters

$$W(z) = \frac{z^{-1} + 1/2\, z^{-2}}{1 + 1/3\, z^{-1}}$$

This type of digital filter is called IIR filter (infinite impulse response): in fact, the time domain output signal is

$$y(t) = -1/3\, y(t-1) + u(t-1) + 1/2\, u(t-2)$$

infinite long division due to the recursive part

$$W(z) = z^{-1} + \frac{1}{2} z^{-2} + \frac{1}{3} z^{-3}$$

This type of digital filter is called FIR (finite impulse response): infact, the time domain output signal is

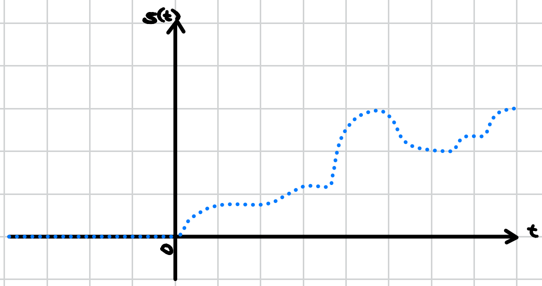
$$y(t) = u(t-1) + \frac{1}{2} u(t-2) + \frac{1}{3} u(t-3) \quad \Rightarrow \quad IR = \{0, 1, 1/2, 1/3, 0, 0 \dots\} \text{ is finite}$$

4. Impulse Response to Transfer function

To make this transformation we need to recall a definition: given a discrete time signal $s(t)$ with $s(t) = 0$ for $t < 0$, the z transform of signal $s(t)$ is defined as

$$\mathcal{Z}[s(t)] = \sum_{t=0}^{+\infty} s(t)\, z^{-t}$$

is a function of z (not of t) and is the transform of a signal (not of a system)



It can be proven that

$$W(z) = \mathcal{Z}[w(t)] = \sum_{t=0}^{+\infty} w(t)\, z^{-t}$$

The transfer function of a system, is the Z-transform of its impulse response. However, in practice this formula cannot be used because

- we need infinite values of IR
- the IR signal must be available noise-free

5. State Space To Impulse Response

Start from the state space system

$$\begin{cases} x(t+1) = Fx(t) + Gu(t) \\ y(t) = Hx(t) + \cancel{Du(t)} \end{cases}$$

Initial conditions: $x(0)=0$ and $y(0)=0$

Run the system starting from the initial condition

$x(1) = Fx(0) + Gu(0) = Gu(0)$ $\qquad$ $y(1) = Hx(1) = HGu(0)$

$x(2) = Fx(1) + Gu(1) = FGu(0) + Gu(1)$ $\qquad$ $y(2) = Hx(2) = HFGu(0) + HGu(1)$

$x(3) = Fx(2) + Gu(2) = F^2Gu(0) + FGu(1) + Gu(2)$ $\qquad$ $y(3) = Hx(3) = HF^2Gu(0) + HFGu(1) + HGu(2)$

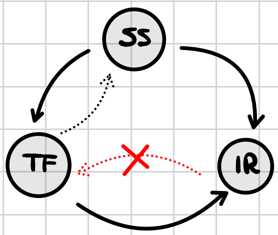
you can get the general rule:

$$y(t) = 0\,u(t) + HG\,u(t-1) + HFG\,u(t-2) + HF^2G\,u(t-3) + \dots$$

So $IR = \{0, HG, HFG, HF^2G \dots\}$. The general formula for the IR is

$$\omega(t) = \begin{cases} 0 & t=0 \\ HF^{t-1}G & t>0 \end{cases}$$

6. Impulse Response To State Space



Moving from IR To SS is the key task of a black box system identification method called Subspace-based State Space System Identification (4SID) method

in MATLAB sys ID Toolbox : > n4Sid

Easy To make an experiment To build an IR

Now we recall the fundamental concept of observability and controllability of a system. given the SS representation

$$\begin{cases} x(t+1) = Fx(t) + Gu(t) \\ y(t) = Hx(t) + \cancel{Du(t)} \end{cases}$$

The system is fully observable from the output $y(t)$ if and only if the observability matrix

$$O = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix}$$

is full-rank ($RANK(O) = n$)

Observability depends only of F and H so it's a property related to state → output relationship: by watching the output signal $y(t)$, we can observe the full state $x(t)$

The system is fully controllable from input To state iff the controllability matrix

$$R = \begin{bmatrix} G & FG \dots F^{n-1}G \end{bmatrix}$$

is full-rank ($RANK(R) = n$)

Controllability depends only on G and F, so it's a property of input → state relationship: by moving the input, we can control the full state $x(t)$

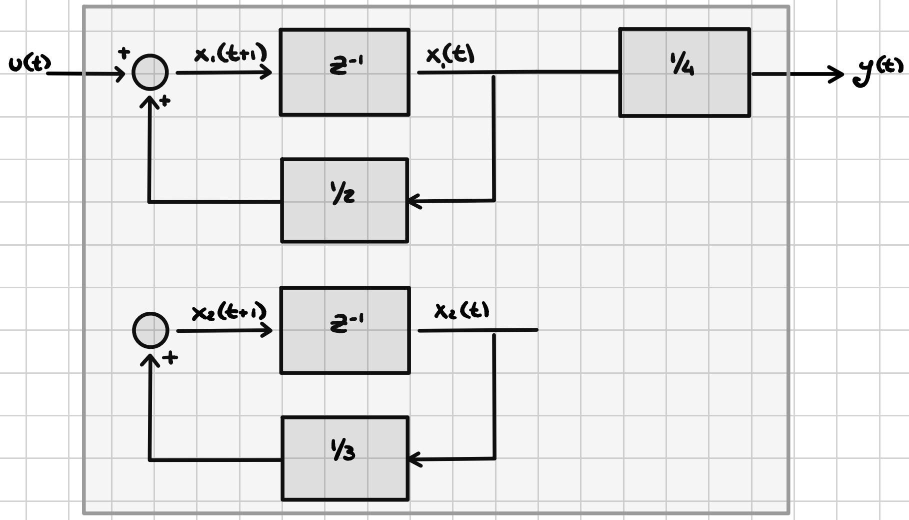
example:

$$S: \begin{cases} x_1(t+1) = \frac{1}{2} x_1(t) + u(t) \\ x_2(t+1) = \frac{1}{3} x_2(t) \\ y(t) = \frac{1}{4} x_1(t) \end{cases}$$

$n = 2$ $\quad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ SISO

$$F = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \quad G = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad H = \begin{bmatrix} \frac{1}{4} & 0 \end{bmatrix} \quad D = 0$$

Block - scheme



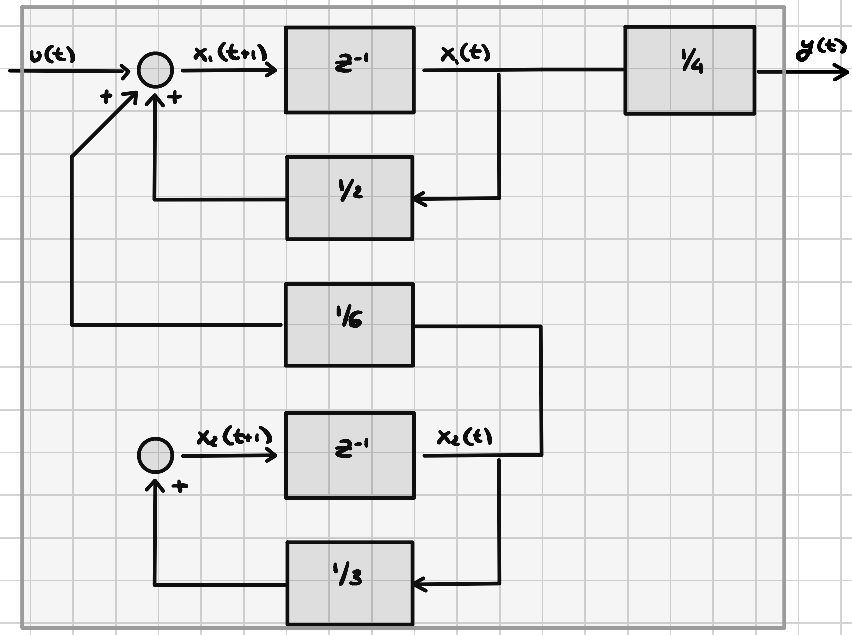
observability

$$O = \begin{bmatrix} H \\ HF \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{8} & 0 \end{bmatrix}$$

RANK(O) = 1 < 2 ⇒ not full rank ⇒ not fully observable

Modified system

$$S: \begin{cases} x_1(t+1) = \frac{1}{2} x_1(t) + u(t) + \frac{1}{6} x_2(t) \\ x_2(t+1) = \frac{1}{3} x_2(t) \\ y(t) = \frac{1}{4} x_1(t) \end{cases}$$



$$F = \begin{bmatrix} \frac{1}{2} & \frac{1}{6} \\ 0 & \frac{1}{3} \end{bmatrix} \Rightarrow O = \begin{bmatrix} \frac{1}{4} & 0 \\ \frac{1}{8} & \frac{1}{24} \end{bmatrix}$$

full-rank ⇒ s.s. is fully observable

example:

$$S: \begin{cases} x_1(t+1) = \frac{1}{2} x_1(t) \\ x_2(t+1) = \frac{1}{3} x_2(t) + u(t) \\ y(t) = \frac{1}{4} x(t) \end{cases}$$

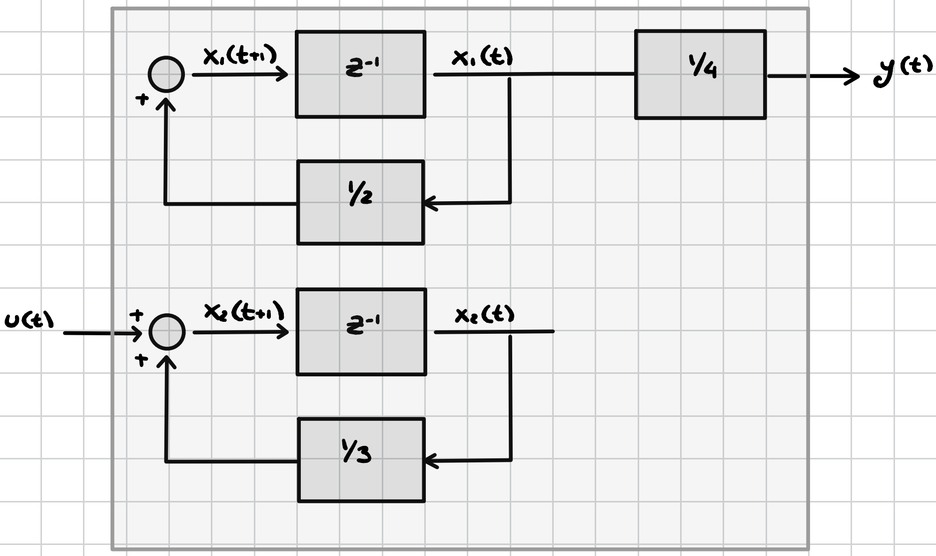
$$F = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad H = \begin{bmatrix} \frac{1}{4} & 0 \end{bmatrix}$$

Check the controllability

$$R = \begin{bmatrix} G & FG \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & \frac{1}{3} \end{bmatrix}$$

$RANK(R) = 1 < 2$ not fully controllable

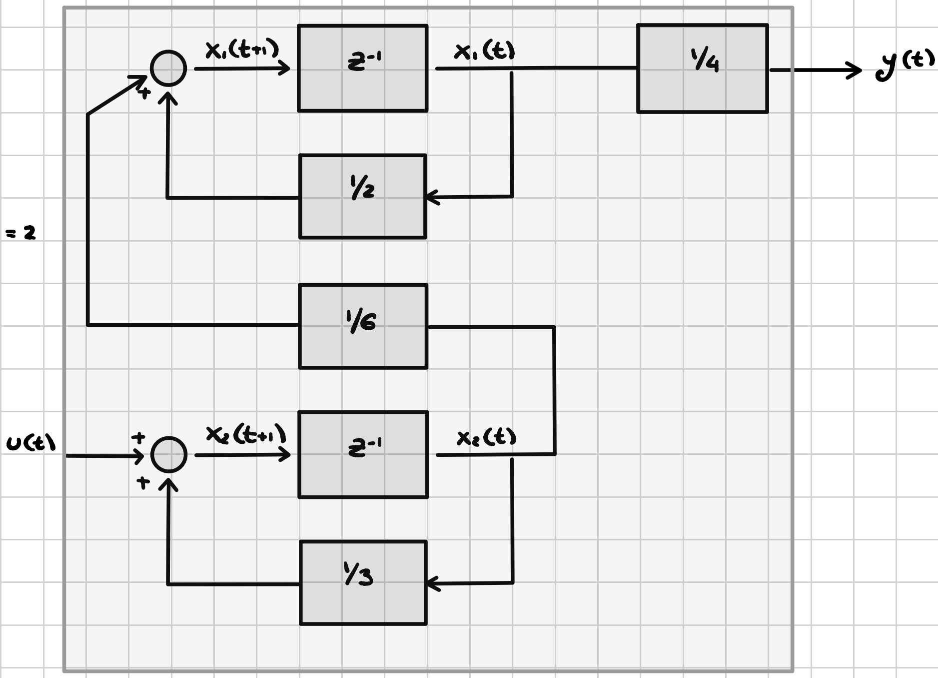
x_1 is not controllable



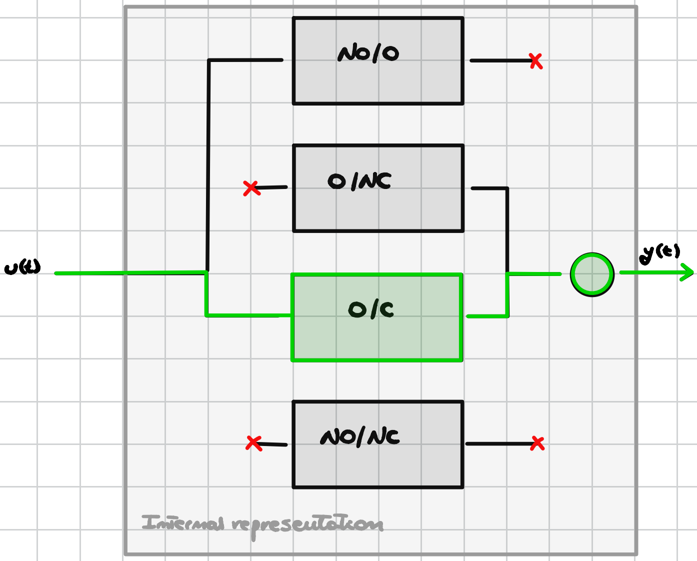
Modified system

$$S: \begin{cases} x_1(t+1) = \frac{1}{2} x_1(t) + \frac{1}{6} x_2(t) \\ x_2(t+1) = \frac{1}{3} x_2(t) + u(t) \\ y(t) = \frac{1}{4} x(t) \end{cases}$$

$$F = \begin{bmatrix} \frac{1}{2} & \frac{1}{6} \\ 0 & \frac{1}{3} \end{bmatrix} \Rightarrow R = \begin{bmatrix} 0 & \frac{1}{6} \\ 1 & \frac{1}{3} \end{bmatrix} \Rightarrow RANK(R) = 2$$



Any state space system can be divided into 4 sub-systems
- not observable / controllable (NO/C)
- observable / not controllable (O/NC)
- observable / controllable (O/C)
- not observable / not controllable (NO/NC)



The external I/O representation can only describe the controllable / observable subsystem of the internal representation. So, all the other subsystem are hidden from an I/O perspective

Only the O/C can be seen from the external point of view

DEF HANK MATRIX of order n build from the impulse response $IR = \{ \omega(0), \omega(1), \dots \}$

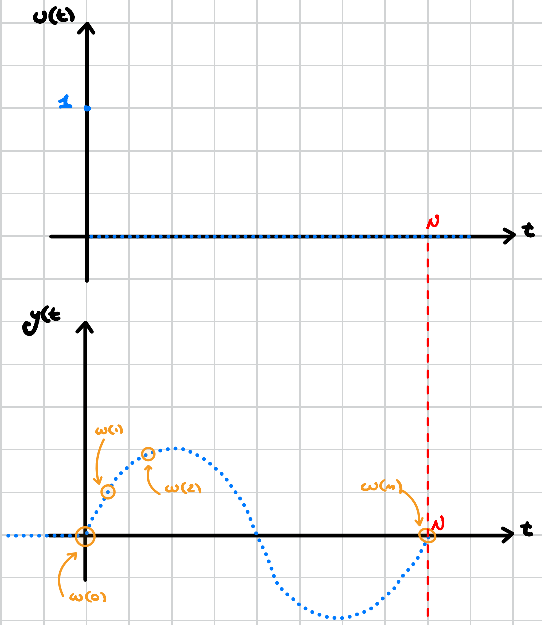
$$H_n = \begin{bmatrix} \omega(1) & \omega(2) & \dots & \omega(n) \\ \omega(2) & \ddots & & \omega(n+1) \\ \vdots & & \ddots & \vdots \\ \omega(n) & & \dots & \omega(2n+1) \end{bmatrix}$$

we start from $\omega(1)$ (and not $\omega(0)$). It's a square matrix $n \times n$. All the counter diagonal contains the same element and the last element is $\omega(2n-1)$

Recall the transformation $SS \to IR$: $\omega(t) = HF^{t-1}G$ for $t>0$. Replace this formula into H_n, so

$$H_n = \begin{bmatrix} HG & HFG & HF^{n-1}G \\ HFG & \ddots & HF^{n}G \\ HF^{n-1}G & \dots & HF^{2n-2}G \end{bmatrix} = \underbrace{\begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix}}_{\text{observability matrix } O} \underbrace{\begin{bmatrix} G & FG \dots & F^{n-1}G \end{bmatrix}}_{\text{controllability matrix } R} \Rightarrow H_n = OR$$

Now we can present the 4SID algorithm to obtain an estimated SS model in a black box way starting from an experimental measured truncated IR of the system



Dataset collected (Truncated) : $\{ \omega(0), \dots, \omega(N) \}$

We start with the assumption that the IR is measured noise-free (Ideal situation)

4SID procedure, starting from a noise-free experiment

1. build the Hankel matrix in increasing order and check the rank

$H_1 = [\,\omega(1)\,]$ ⇒ RANK = 1

$H_2 = \begin{bmatrix} \omega(1) & \omega(2) \\ \omega(2) & \omega(3) \end{bmatrix}$ ⇒ RANK = 2

⋮

$H_n = [\,\dots\,]$ ⇒ RANK = n

$H_{n+1} = [\,\dots\,]$ ⇒ RANK = n STOP ⇒ we found the 1° Henkel matrix not full-rank

2. Take H_{n+1} (This is an $(n+1)\times(n+1)$ matrix of rank = n). We can factorize H_{n+1} into two rectangular matrices of size $(n+1)\times n$ and $n\times(n+1)$:

$$H_{n+1} = \underbrace{\begin{bmatrix} O_{n+1} \end{bmatrix}}_{(n+1)\times n}\underbrace{\begin{bmatrix} R_{n+1} \end{bmatrix}}_{n\times(n+1)} \Rightarrow H_{n+1} = O_{n+1}R_{n+1}$$

we consider this matrix as the extended controllability matrix

we consider this matrix as the extended (n+1) observability matrix

3. Estimate $\hat{F}, \hat{G}, \hat{H}$ from O_{n+1} and R_{n+1}

$$O_{n+1} = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n} \end{bmatrix} \qquad R_{n+1} = \begin{bmatrix} G & FG & \dots & F^{n}G \end{bmatrix}$$

$\hat{G} = R_{n+1}(:;1)$ any row / first col

$\hat{H} = O_{n+1}(1;:)$ first row / any cols

To estimate $\hat{F}$ consider, for example O_{n+1} (The same can be done also from R_{n+1})

$$O_{n+1} = \begin{bmatrix} H \\ HF \\ HF^2 \\ \vdots \\ HF^{n-1} \\ HF^{n} \end{bmatrix}$$

$\rightarrow O_1 = O_{n+1}(1:n;:)$

$\rightarrow O_2 = O_{n+1}(2:n+1;:)$

O_1 and O_2 are square matrices linked by the "shift invariance property"

$$O_2 = O_1 F$$

Since O_1 is square and invertible. So

$$\hat{F} = O_1^{-1}O_2$$

We have estimated a BB SS model $\{\hat{F}, \hat{G}, \hat{H}\}$ starting from a noise-free experimentally measured impulse response

This method is constructive and non parametric. Remind that a typical parametric estimation algorithm from data is based on these 4 items:

1) Collect a Dataset:

$$\{u(1), u(2), \dots, u(N)\}\{y(1), y(2), \dots, y(N)\}$$

2) Choose a parametric model:

$$M(\vartheta):\ y(t) = W(z;\vartheta)\,u(t)$$

with $W(z,\vartheta)$ is the Transfer function and ϑ is the vector of coefficients of NUM/DEN

3) Define the performance index:

$$J(\vartheta) = 1/N \sum_1^N (y(t) - W(z;\vartheta)\,u(t))^2$$

eg. sample variance of the output error made by the model. We can "sort" the quality of the model (eg. if $J(\vartheta_1) < J(\vartheta_2)$ so $M(\vartheta_1)$ is better than $M(\vartheta_2)$)

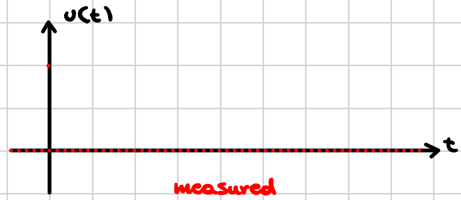
4) Optimization step: we minimize $J(\vartheta)$ with respect to ϑ

$$\hat{\vartheta}_N = \arg\min_\vartheta J(\vartheta)$$

so $M(\hat{\vartheta})$ is the best model

4SID for Noise-free data was known since '60, useless since it is not robust with respect to Noise on $\omega(t)$.
The method was rediscovered in '90 when a fundamental result of numerical algebra was developed:
Singular Value Decomposition (SVD) [MATLAB was build in the same period]

Now we move to the real problem: we assume that the IR is measured with Noise



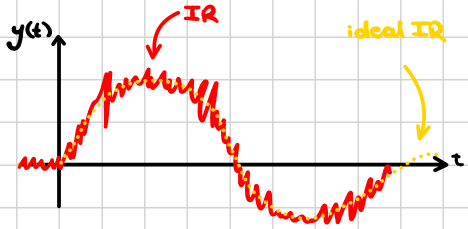
$$\tilde{\omega}(t) = \omega(t) + \eta(t) \qquad \forall t = 0,1,2 \dots N$$

$N \approx 500 - 10.000$

$\tilde{\omega}(t)$: noisy measured signal

$\omega(t)$: ideal non noisy signal

$\eta(t)$: noise



Dataset Available: $\{\tilde{\omega}(0), \tilde{\omega}(1), \dots, \tilde{\omega}(N)\}$

We see the algorithm that start from Measured IR, but 4SID can be extended to any input dataset

4SID Algorithm in case of Noisy IR dataset

1) Build the Henkel matrix using the full dataset

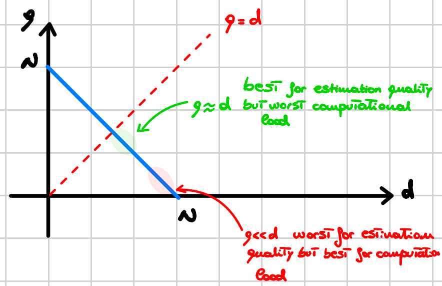
start from $\tilde{\omega}(1)$ (not from $\tilde{\omega}(0)$) assuming the strictly proper system

build using noisy data

$$\tilde{H}_{qd} = \begin{bmatrix} \tilde{\omega}(1) & \tilde{\omega}(2) & \dots & \tilde{\omega}(d) \\ \tilde{\omega}(2) & \dots & \dots & \tilde{\omega}(d+1) \\ \vdots & & & \\ \tilde{\omega}(q) & & & \tilde{\omega}(q+d-1) \end{bmatrix}$$

rectangular $q \times d$ (with $q < d$)

$q+d-1 = N \Rightarrow$ we are using the whole dataset



Assuming $q < d$ by construction and $q+d-1 = N \Rightarrow q = N+1-d$

Rule of Thumb: $q \geq \frac{1}{2} d$ (eg. $N = 1000 \Rightarrow (q = 350 \;\; d = 651), (q = 400, \; d = 601)$, $(q = 450, \; d = 551)$ ... these are good choices, you get very similar final results).

2) SVD of $\tilde{H}_{qd}$

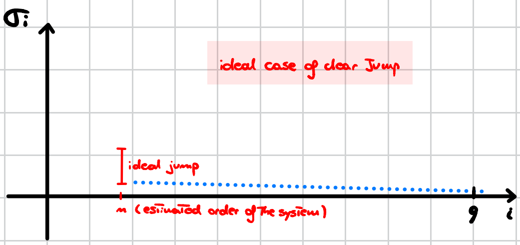
$$\tilde{H}_{qd} = \tilde{U}_{q \times q} \cdot \tilde{S}_{q \times d} \cdot \tilde{V}^T_{d \times d}$$

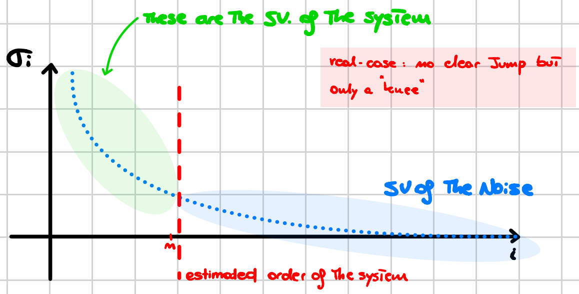
- $\tilde{U}$ and $\tilde{V}$ are square and unitary (M is unitary if $\det(M) = 1$ and $M^{-1} = M^T$) matrices
- $\tilde{S} = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_q & 0 \end{bmatrix}$ is a diagonal rectangular matrix where $\{\sigma_1, \sigma_2, \dots, \sigma_q\}$ are called singular values of $\tilde{H}_{qd}$. They are real positive numbers sorted in decreasing order ($\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \dots \geq \sigma_q$)

SVD is a sort of diagonalization of a rectangular matrix. Singular values are sort of eigen values for a rectangular matrix. Notice that if H is rectangular: $SV(H) = \sqrt{EIG(H \cdot H^T)}$

SVD is well implemented in MATLAB: `[U,S,V] = svd(H)` (like in many other numerical algebra software)

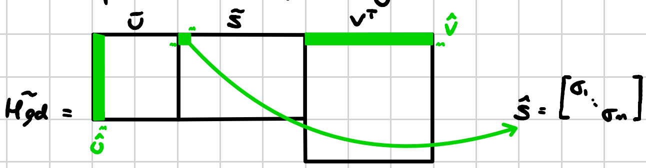
3) Plot the singular values and cut the system part of the matrix





We have made a system-Noise separation. Typically $n \ll q$ (eg $q = 400$, $n = 8$)

Now we can complete the cut by cutting the $\tilde{U}, \tilde{S}, \tilde{V}$ matrices:



$\hat{S} = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}$

Define a new matrix $q \times d$ like the original

$\hat{H}_{qd} = \hat{U}\hat{S}\hat{V}^T$

$\tilde{H}_{qd} = \hat{H}_{qd} + H_{res_{qd}}$

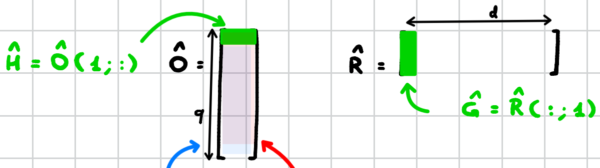
original rank = q — clean rank = n — residual noise matrix rank = q

4) Estimation of $\hat{F}, \hat{G}, \hat{H}$ starting from the clean matrix $\hat{H}_{qd}$

$$\hat{H}_{qd} = \hat{U}\hat{S}\hat{V}^T = \hat{U}\hat{S}^{1/2}\hat{S}^{1/2}\hat{V}^T \quad \text{with} \quad \hat{S}^{1/2} = \begin{bmatrix} \sqrt{\sigma_1} & & 0 \\ 0 & \sqrt{\sigma_2} & 0 \\ 0 & \dots & \ddots \end{bmatrix}$$

We can define $\hat{O} = \hat{U}\hat{S}^{1/2}$ and $\hat{R} = \hat{S}^{1/2}\hat{V}^T$ so it's possible to rewrite $\hat{H}_{eq} = \hat{O}\hat{R}$

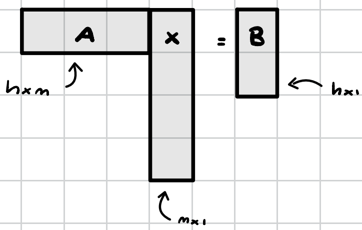
$\hat{O}$ and $\hat{R}$ can been seen as extended observability and controllability matrices



$O_2 = \hat{O}(2{:}q;:)$ — $O_1 = \hat{O}(1{:}q{-}1;:)$ ⇒ shift invariance property of observability matrix $O_1 \cdot F = O_2$ ⇒ $\hat{F} = \cancel{O_1^{-1}} O_2$ not a square matrix!

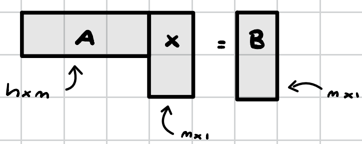
Solution of a linear system $Ax = B$

1)

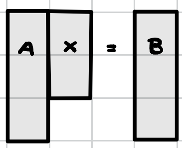


⇒ $h < n$ ⇒ less equation than variables so the system is under-determined and ∞ solutions

2)



⇒ $h = m$ ⇒ "square system" of m equation and m variables, so only 1 solution (if A is invertible): $X = A^{-1}B$



⇒ $h \gg n$ ⇒ more equation than variables (The system is over-constrained so there are no solutions

We can solve this system in approximate "least squares"

$$Ax = B \Rightarrow \underbrace{A^T A}_{\text{square}} x = A^T B \Rightarrow \hat{x} = (A^T A)^{-1} A^T B$$

is called "Pseudo-inverse" (Surrogate of the inverse for the rectangular A matrix)

So going back to our case, we can solve it using the point 3

$$O_1 F = O_2 \Rightarrow O_1^T O_1 F = O_1^T O_2 \Rightarrow \hat{F} = (O_1^T O_1)^{-1} O_1^T O_2$$

We have found the state-space model given by $\{\hat{F}, \hat{G}, \hat{H}\}$

Starting from a measured Noisy IR $\{ \omega(1), \dots, \omega(N) \}$ we have estimated (in a Black-Box way) a SS model of the system

$$u(t) \rightarrow \boxed{\begin{cases} x(t+1) = \hat{F}x(t) + \hat{G}u(t) \\ y(t) = \hat{H}x(t) \end{cases} \Downarrow \; \hat{W}(z) = \hat{H}(zI - \hat{F})^{-1}\hat{G}} \rightarrow y(t)$$

The Optimality of 4SID model

The method is optimal since SVD makes an optimal rank reduction. Notice that the rank reduction of a matrix can be done in infinite way

Example:

$$\begin{bmatrix} 2 & 5 & 3 & 6 & 5 \\ 5 & 3 & 6 & 5 & 7 \\ 3 & 6 & 5 & 7 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 5 & 3 & 6 & 5 \\ 5 & 2 & 6 & 5 & 7 \\ 3 & 6 & 5 & 7 & 1 \end{bmatrix}$$

full rank = 3   rank = 2   residual rank = 3

Is this an optimal rank reduction? No

$$\underbrace{\tilde{H}_{qd}}_{\text{rank}=q} = \underbrace{\hat{H}_{qd}}_{\text{rank}=n} + \underbrace{H_{res_{qd}}}_{\text{rank}=q} \qquad n \ll q$$

The goal of rank reduction $(q \rightarrow n)$: we have to minimize the residual matrix $H_{res_{qd}}$; we have to reduce rank by discarding the minimum amount of information. This is a minimization with respect to the Frobenius Norm

$$| H_{RES_{qd}} |_F = \sqrt{\sum_{ij} (H^{ij}_{RES_{qd}})^2}$$

Optimality comes from the sorting of singular values

SVD was the turning point in machine learning because it allows a numerically efficient way to solve:
- Compression of information
- Separation of signal and noise
- Order reduction of systems

3 different perspective of the same problem

Consider the system with the following SS representation

$$F = \begin{bmatrix} 1/2 & 0 \\ 1 & 1/4 \end{bmatrix} \quad G = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad H = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad D = 0 \longleftarrow \text{Strictly proper system}$$

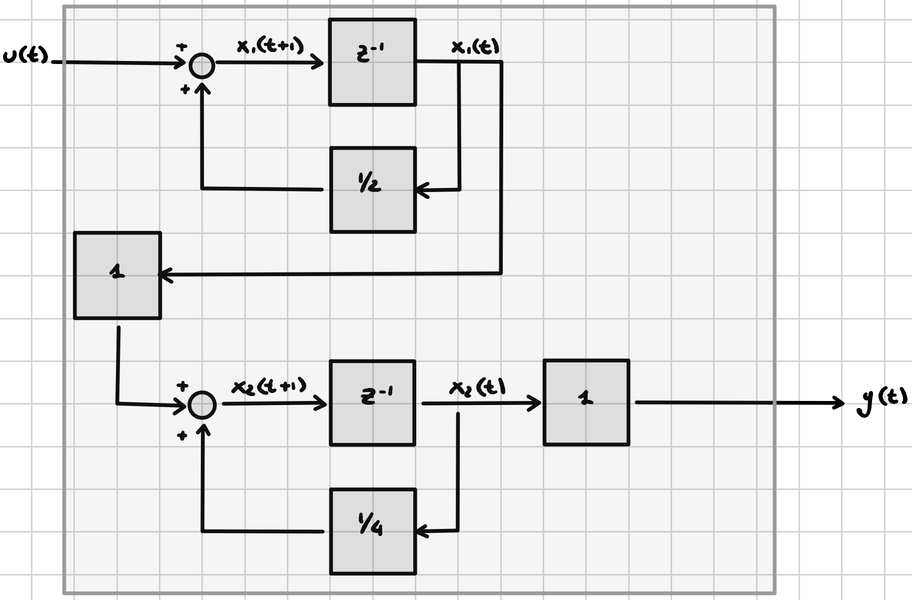
SISO

└ Triangular matrix $\Rightarrow EIG(F) = \{1/2, 1/4\} \Rightarrow$ asymptotically stable ($|EIG| < 1$)

1) Write the time domain state space equation

$$\begin{cases} x_1(t+1) = 1/2\, x_1(t) + 0\, x_2(t) + u(t) \\ x_2(t+1) = x_1(t) + 1/4\, x_2(t) + 0\, u(t) \\ y(t) = 0\, x_1(t) + x_2(t) + 0\, u(t) \end{cases}$$

2) Draw the Black-Box model of the SS system and check the observability and controllability



Formal check:

$$O = \begin{bmatrix} H \\ HF \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1/4 \end{bmatrix} \Rightarrow \text{rank} = 2 = n \;\checkmark$$

$$R = \begin{bmatrix} G & FG \end{bmatrix} = \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix} \Rightarrow \text{rank} = 2 = n \;\checkmark$$

Compute the extended R and O

$$O_3 = \begin{bmatrix} 0 & 1 \\ 1/4 & 1/4 \\ 3/4 & 1/16 \end{bmatrix} \qquad R_3 = \begin{bmatrix} 1 & 1/2 & 1/4 \\ 0 & 1 & 3/4 \end{bmatrix}$$

3) Compute the Transfer function from $u(t)$ to $y(t)$

We can do it by direct manipulation of SS equation

$$x_1(t+1) = \frac{1}{2} x_1(t) + u(t) \Rightarrow x_1(t) = \frac{1}{z - 1/2} u(t)$$

$$x_2(t+1) = x_1(t) + 1/4\, x_2(t) \Rightarrow x_2(t) = \frac{1}{(z-1/2)(z-1/4)} u(t)$$

So $y(t) = x_2(t) = \frac{1}{(z-1/2)(z-1/4)} u(t)$. The Transfer function has two poles $(1/2, 1/4)$ that are the EIG(F)

Using the formula

$$W(z) = H(zI-F)^{-1}G = [0 \;\; 1]\left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 1/2 & 0 \\ 1 & 1/4 \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \dots = \frac{1}{(z-1/2)(z-1/4)}$$

4) Compute the Impulse Response up to $\omega(5)$

$$W(z) = \frac{1}{(z-1/2)(z-1/4)} = \frac{1}{z^2 - 3/4\, z + 1/8} = \frac{z^{-2}}{1 - 3/4\, z^{-1} + 1/8\, z^{-2}}$$

$$\begin{array}{l|l} z^{-2} & 1 - 3/4\, z^{-1} + 1/8\, z^{-2} \\ \hline -\;z^{-2} + 3/4\, z^{-3} - 1/8\, z^{-4} & z^{-2} + 3/4\, z^{-3} + 7/16\, z^{-4} + 15/64\, z^{-5} \\ \hline 3/4\, z^{-3} - 1/8\, z^{-4} & \\ \vdots & \end{array}$$

So we use the long division we found

$\omega(0) = 0$ $\quad$ $\omega(3) = 3/4$

$\omega(1) = 0$ $\quad$ $\omega(4) = 7/16$

$\omega(2) = 1$ $\quad$ $\omega(5) = 15/64$

The system has 2 delay steps

5) Build the Henkel matrix and "find" the system order

$$\left.\begin{aligned}
H_1 &= \begin{bmatrix} 0 \end{bmatrix} \\
H_2 &= \begin{bmatrix} 0 & 1 \\ 1 & 3/4 \end{bmatrix} \qquad \text{Rank} = 2 \\
H_3 &= \begin{bmatrix} 0 & 1 & 3/4 \\ 1 & 3/4 & 7/16 \\ 3/4 & 7/16 & 15/64 \end{bmatrix} \qquad \text{Rank} = 2!
\end{aligned}\right\} n = 2$$