2 - Black Box System Identification using classic parametric approach for I/O Transfer function model in frequency domain - Frequency domain system identification

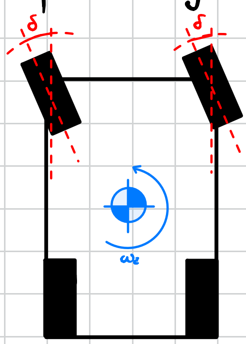
the method is very intuitive, very used in closed loop control application using the classic 4 steps ML parametric methods :

1. Experiment Design, data collection and data preprocessing
2. Selection of a family of a parametric models $M(\vartheta)$, with ϑ parameter vector
3. Definition of performance index $J(\vartheta)$
4. Optimization : $\hat{\vartheta} = \mathrm{argmin}_{\vartheta} \, J(\vartheta) \Rightarrow M(\hat{\vartheta})$ is the optimal model

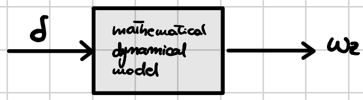
The general intuitive idea of the method is:
- making a set of simple sinusoid / frequency / Tone excitation experiment
- from each experiment, extract and estimate a single point of the frequency response of the system
- fit the measured frequency response with the frequency response of the of the system

This is a classic Supervised Learning Technique applied in frequency domain

Example: I/O dynamics estimation of a Steer-to-Yawrate in a car

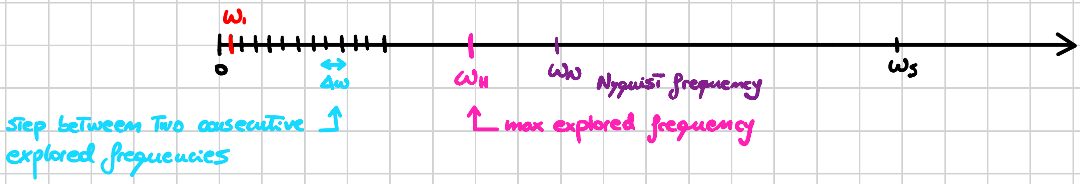


ω_z : rotational speed of the car around its vertical axis (Yaw-rate)



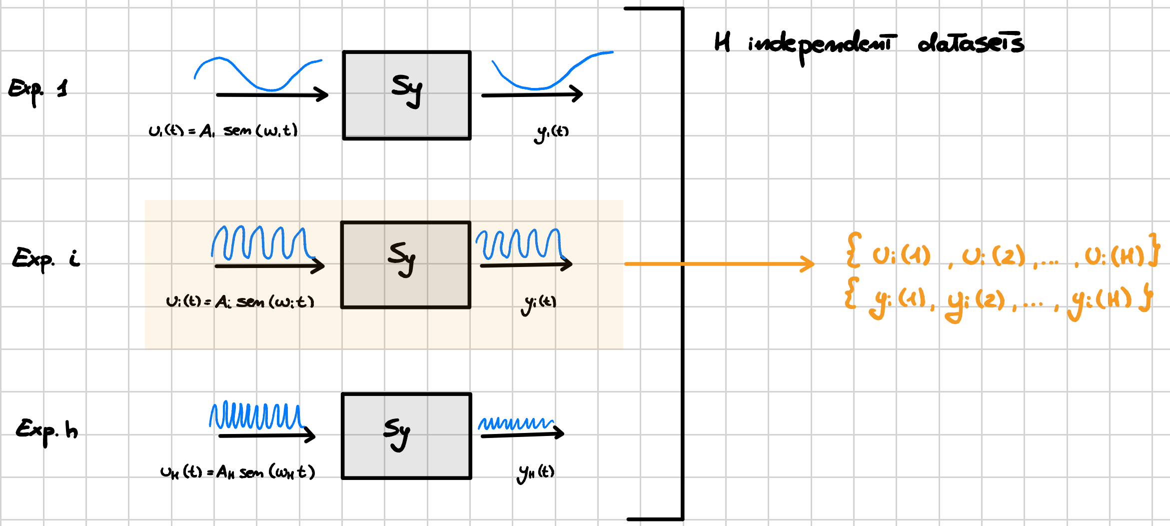
This I/O relationship is very useful to design electronic stability control (ESC or ESP) and L2 + autonomous cars (Lateral dynamical control)

In the experiment design the first step is to select a set of excitation frequency



Usually $\Delta\omega$ is constant : $\{\omega_1, \dots, \omega_i, \dots, \omega_H\}$ are uniformly distributed and in general $\omega_H \ll \omega_N$ (Nyquist frequency)

Next step : perform H independent I/O experiment on the system



Choice of amplitude $\{A_1, \dots, A_H\}$: the amplitudes of the H sinusoides can be identical, but usually they have decreasing values the meet actuation limits (eg. Power limitation of the steer activator)

Example: Steer input $\delta(t)$ in a car; the requested steering Torque is proportional to δ: $T(t) = k\,\delta(t)$

- Steering power requested: Torque · rotational-speed
- power: $k\,\delta(t)\dot\delta(t)$

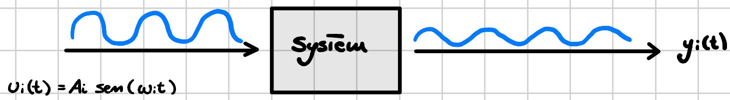
Assume the $\delta(t) = A_i \operatorname{sen}(\omega_i t)$, the steering power becomes

$$k A_i \operatorname{sem}(\omega_i t)\,\omega_i A_i \cos(\omega_i t) = \underbrace{k A_i^2 \omega_i}_{\text{Amplitude of the power signal}} \cdot \operatorname{sem}(\omega_i t)\cos(\omega_i t)$$

If there is a power limitation:

$$k A_i^2 \omega_i = \text{constant} \Rightarrow A_i = \sqrt{\frac{\text{constant}}{k\,\omega_i}}$$ Amplitude limitation

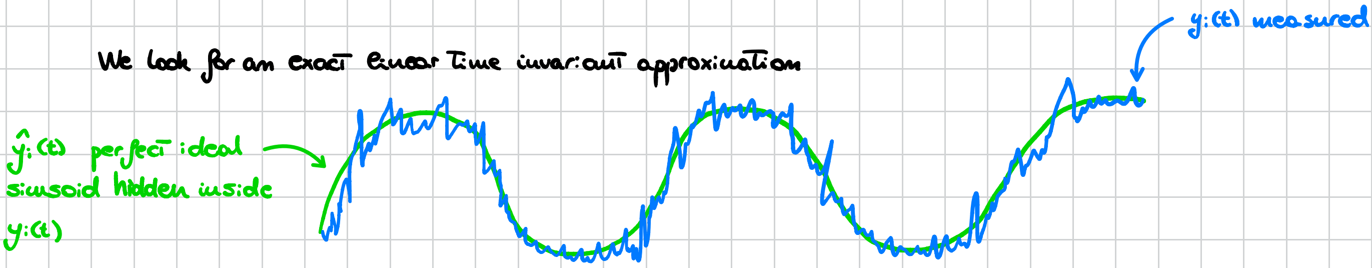
Let's focus now on the generic i-th experiment



We are assuming that the system is LTI: for LTI system, if the input is a sinusoid of frequency ω_i, the output must be a sinusoidal of frequency ω_i (with different amplitude and phase but with only frequency ω_i. Does not happen if system is not LTI, it can be used as "linearity test". However, in real application, $y_i(t)$ is never exactly a perfect Sinusoid because:

- noise on the output (measured noise)
- noise in the system (internal noise)
- small non-linear effects (that we neglect)

We look for an exact linear time invariant approximation



$\hat{y}_i(t)$ is the noise cleaned version of the measured $y_i(t)$; the problem is to find $\hat{y}_i(t)$ (pre-processing problem, is a simple model identification problem)

Dataset: $\{y_i(1), \dots, y_i(N)\}$

Model for $y_i(t)$:

$$\hat{y}_i(t) = \begin{cases} B_i \sin(\omega_i t + \varphi_i) \\ a_i \operatorname{sen}(\omega_i) + b_i \cos(\omega_i t) \end{cases}$$

much better for system identification because is linear with respect to the two parameters

We can estimate a_i and b_i using classic parametric supervised learning.

Performance index:

$$J_N(a,b) = 1/N \sum_{t=1}^{N} \Big(\underbrace{y_i(t)}_{\text{True measured output}} - \underbrace{(a_i \operatorname{sen}(\omega_i t) + b_i \cos(\omega_i t))}_{\text{modeled cleaned output}}\Big)^2$$

Optimization of performance index: easy since its a quadratic function

$$\hat{a}_i, \hat{b}_i = \operatorname{argmin} J_N(a_i, b_i) = \begin{cases} \partial/\partial a_i\, J_N = 2/N \sum_1^N - \operatorname{sen}(\omega_i t)\big(y_i(t) - a_i \operatorname{sen}(\omega_i t) + b_i \cos(\omega_i t)\big) = 0 \\ \partial/\partial b_i\, J_N = 2/N \sum_1^N - \cos(\omega_i t)\big(y_i(t) - a_i \operatorname{sen}(\omega_i t) + b_i \cos(\omega_i t)\big) = 0 \end{cases}$$

It's a 2x2 linear system: classic "least squares" solution

$$\begin{bmatrix} a_i \\ b_i \end{bmatrix} = \begin{bmatrix} \sum \mathrm{sen}^2(\omega_i t) & \sum \mathrm{sen}(\omega_i t)\cos(\omega_i t) \\ \sum \mathrm{sen}(\omega_i t)\cos(\omega_i t) & \sum \cos^2(\omega_i t) \end{bmatrix}^{-1} \cdot \begin{bmatrix} \sum y_i(t)\,\mathrm{sen}(\omega_i) \\ \sum y_i(t)\cos(\omega_i) \end{bmatrix}$$

Now we prefer to go back to the Amplitude-Phase representation of the Sinusoid ($\hat{y}(t) = B_i\,\mathrm{sen}(\omega_i t + \varphi_i)$)

$$\hat{y}_i(t) = \hat{a}_i\,\mathrm{sen}(\omega_i t) + \hat{b}_i\cos(\omega_i t) = \hat{B}_i\,\mathrm{sen}(\omega_i t + \hat{\varphi}_i)$$
$$= \hat{B}_i\,\mathrm{sen}(\omega_i t)\cos(\hat{\varphi}_i) + \hat{B}_i\cos(\omega_i t)\,\mathrm{sen}(\hat{\varphi}_i)$$

By watching the two equations:

$$\begin{cases} \hat{B}_i\cos(\hat{\varphi}_i) = \hat{a}_i \\ \hat{B}_i\,\mathrm{sen}(\hat{\varphi}_i) = \hat{b}_i \end{cases} \qquad \frac{\hat{a}_i}{\hat{b}_i} = \frac{\cos\hat{\varphi}_i}{\mathrm{sen}\,\hat{\varphi}_i} \Rightarrow \boxed{\begin{aligned} \hat{\varphi}_i &= \arctan\left(\hat{b}_i/\hat{a}_i\right) \\ \hat{B}_i &= \frac{1}{2}\left(\frac{\hat{a}_i}{\cos\hat{\varphi}_i} + \frac{\hat{b}_i}{\mathrm{sen}\,\hat{\varphi}_i}\right) \end{aligned}}$$

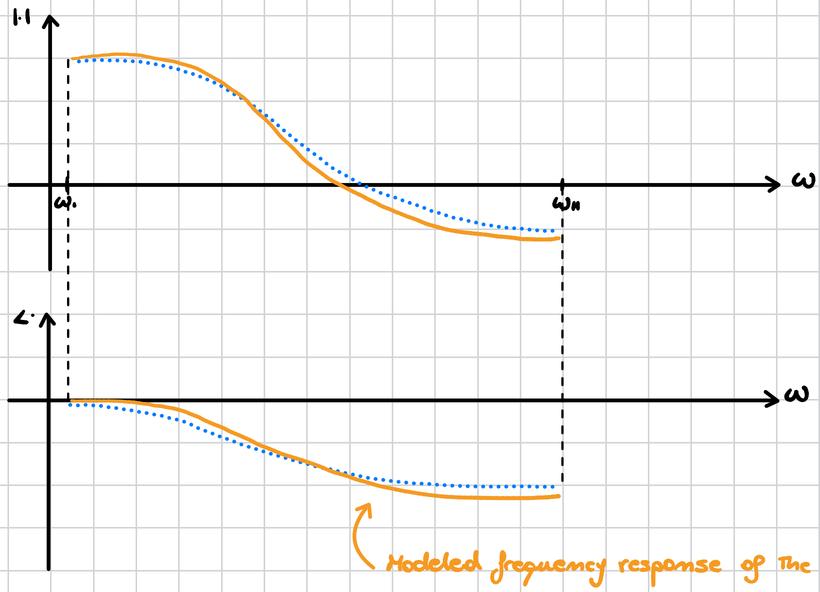
$y_i(t)$ can be approximated $\hat{y}_i(t) = \hat{B}_i\,\mathrm{sen}(\omega_i t + \hat{\varphi}_i)$. In practice we have estimated 1 point of the frequency response of the system

$$\text{input: } A_i\,\mathrm{sen}(\omega_i t + \varphi_i) \longrightarrow \hat{B}_i\,\mathrm{sen}(\omega_i t + \hat{\varphi}_i) \ : \text{output}$$

The amplitude gain is $\hat{B}_i/A_i$ and the phase shift is $\hat{\varphi}_i - 0 = \hat{\varphi}_i$. The frequency response of the system at ω_i is $\hat{B}_i/A_i \cdot e^{j\hat{\varphi}_i}$

By repeating this procedure H times

$$\left.\begin{aligned} \{\hat{B}_1, \hat{\varphi}_1\} &= \hat{B}_1/A_1\, e^{j\hat{\varphi}_1} \\ \{\hat{B}_2, \hat{\varphi}_2\} &= \hat{B}_2/A_2\, e^{j\hat{\varphi}_2} \\ \{\hat{B}_H, \hat{\varphi}_H\} &= \hat{B}_H/A_H\, e^{j\hat{\varphi}_H} \end{aligned}\right\}$$ H complex numbers representing H estimated values of the frequency response of the system



$$u(t) \longrightarrow \boxed{W(z)} \longrightarrow y(t)$$

The frequency response of the TF is

$$W(e^{j\omega}) = |W(e^{j\omega})| \quad \text{amplitude}$$
$$= \angle W(e^{j\omega}) \quad \text{phase}$$

Modeled frequency response of the optimal model $\hat{W}(e^{j\omega}; \hat{\vartheta}_H)$

Step 2: Select a parametric model $M(\vartheta)$ of the system

$$M(\vartheta) = W(z;\vartheta) = \frac{b_0 + b_1 z^{-1} + \dots + b_p z^{-p}}{1 + a_1 z^{-1} + \dots + a_m z^{-m}} z^{-1} \qquad \vartheta = [a_1 \dots a_m \; b_0 \dots b_p]^T$$

Step 3: Performance index selection

$$J_H(\vartheta) = 1/H \sum_{i=1}^{H} \left| W(e^{j\omega_i}, \vartheta) - \frac{\hat{B}_i}{A_i} e^{j\hat{\varphi}_i} \right|^2$$

variance of the frequency domain modeling error

Step 4: Optimization

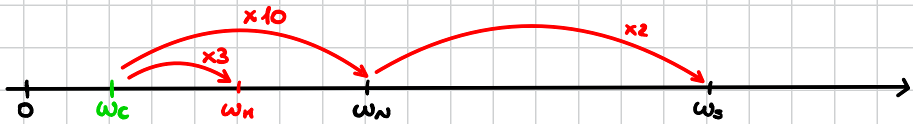
$$\hat{\vartheta}_H = \operatorname{argmin}_{\vartheta} J_H(\vartheta)$$

Usually is a non quadratic optimization problem, so you can use standard tools of iterative optimization (gradient, newton, quasi-newton ...)

$$u(t) \longrightarrow \boxed{\hat{W}(z;\hat{\vartheta}_H)} \longrightarrow y(t)$$

Final estimated model $M(\hat{\vartheta}_H)$

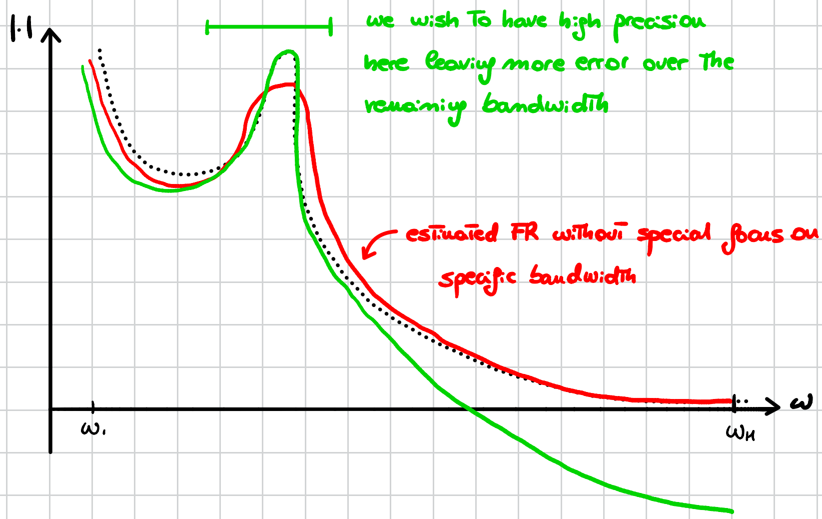
Selection of ω_H: we need to choose a meaningful bandwidth for our model



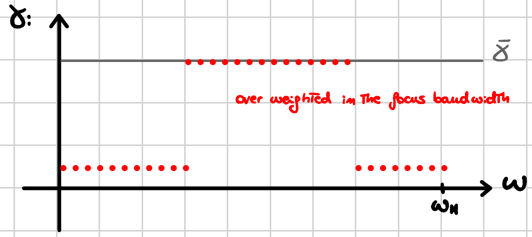
expected closed loop bandwidth of the control system

Example: stability control of a car.
- Control bandwidth = 4Hz $\Rightarrow$ $f_N = 40$ Hz $\quad f_S = 80$ Hz
  $\omega_H = 12$ Hz

In some cases we wish to have higher modeling precision over specific bandwidth between ω_1 and ω_2

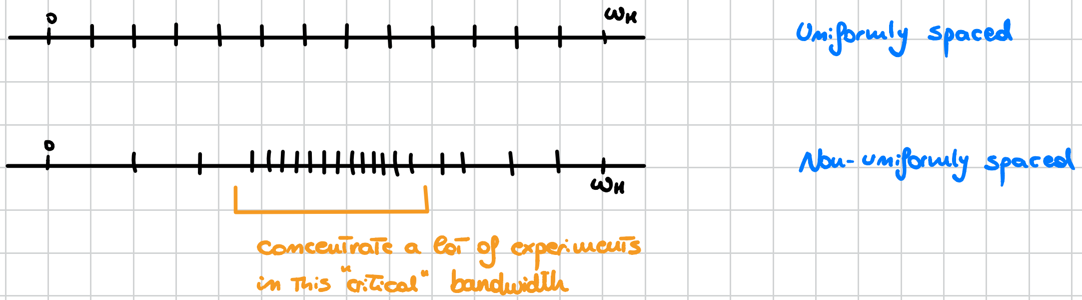


We can manage this special focus with non uniform weights

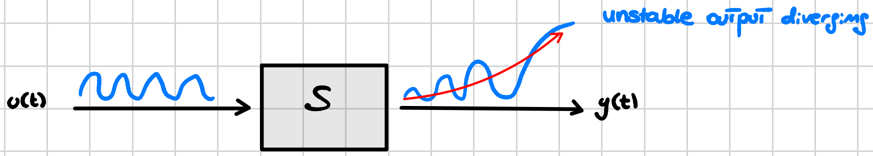


New performance index: $\tilde{J}_H(\vartheta) = 1/H \sum_i^H \gamma_i \left(\left| W(e^{j\omega_i}; \vartheta) - \frac{\hat{B}_i}{A_i} e^{j\hat{\varphi}_i} \right|^2 \right)$

A similar focus affect can be obtained by designing since the beginning the experiment with over-sampling the focus bandwidth

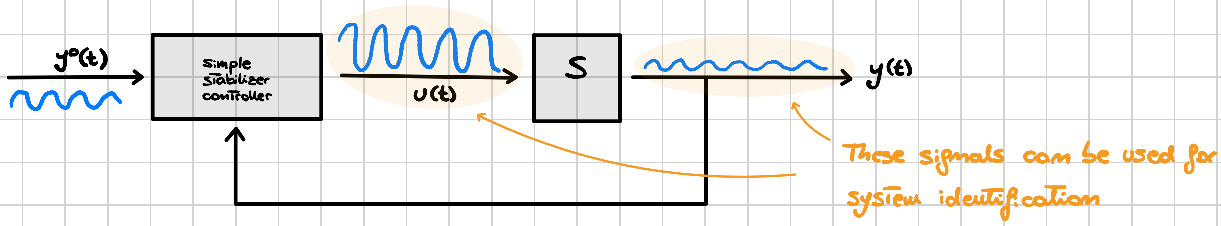


System identification of unstable systems: what happens if we want to collect data with I/O experiment from an unstable system?



This open loop experiment cannot be performed since the system is not stable!

The trick used in this situation is to design a simple "stabilizing" controller and perform a closed loop experiment



This approach is called "closed-loop" system identification.